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Abstract

Full Text

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MATHEMATICS

V. A. ROKHLIN

ON THE CHARACTERISTIC CLASSES OF PONTRYAGIN

(Presented by Academician A. N. Kolmogorov, 15 XI 1956)

1. Formulation of the theorem. Let M^n be a smooth compact orientable manifold; let k be a natural number such that $4k \leq n$, and let χ be a function of an integral argument $i = 1, \dots, n$, defined by the formula $\chi(1) = \dots = \chi(2k) = 2$, $\chi(2k+1) = \dots = \chi(n) = 0$. According to Pontryagin ([1,2]), to the function χ there corresponds a $4k$-dimensional integral characteristic class of the ∇-homologies of the manifold M^n, which we shall denote by $P_{4k}(M^n)$. Only the classes P_{4k} will, in accordance with the terminology that has developed in recent years, be called Pontryagin characteristic classes. All the other characteristic classes defined by Pontryagin in ([1,2]) are expressed, by means of well-known homological operations, in terms of the classes P_{4k} and the Stiefel–Whitney characteristic classes.

As was proved as early as 1950 by Thom ([3]) and Wu ([4]), the Stiefel–Whitney classes are topological invariants of the manifold (i.e. they do not depend on its smooth structure) and can even be expressed in terms of its homological invariants. In contrast to this, the topological invariance of the Pontryagin classes has not been proved up to now and at present constitutes one of the most urgent topological problems. There are only partial results. Thus, Wu ([5]) proved the topological invariance of the classes P_{4k} reduced modulo 3 and 4.

The only Pontryagin class whose topological invariance has been completely proved is the class $P_4(M^4)$ of a four-dimensional manifold M^4; this class was expressed by me ([6]) and by Thom ([7,8]) in terms of invariants of the ∇-homology ring of the manifold M^4. Here is the precise formulation of this theorem.

For $k = 1, 2, \dots$, denote by $\sigma(M^{4k})$ the signature of the quadratic form defined by the Kolmogorov–Alexander multiplication on the $2k$-dimensional group of real ∇-homologies of the oriented manifold M^{4k}, and by $\Sigma_{4k}(M^{4k})$ the $4k$-dimensional class of integral ∇-homologies of the manifold M^{4k} whose index, for any orientation of M^{4k}, is equal to $\sigma(M^{4k})$. It turns out that $P_4(M^4) = 3\Sigma_4(M^4)$.

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This is a special case of the following more general theorem, following from the results of Thom ([8]):

For every $k = 1, 2, \dots$ there exist a polynomial $\varphi_k(x_1, \dots, x_k)$ with integer coefficients and a natural number α_k such that, in the ring of integral ∇-homologies of the manifold M^{4k},

$$\alpha_k \Sigma_{4k}(M^{4k}) = \varphi_k(P_4, \dots, P_{4k}). \tag{1}$$

For details on φ_k and α_k, see Hirzebruch ([9]); we shall note only that φ_k and α_k are uniquely determined by the condition that α_k be relatively prime to

with coefficients $\mathfrak{B}_k$, and that the coefficient β_k with which x_k enters in $\mathfrak{B}_k$ is nonzero. According to the preceding, $\varphi_1(x_1) = x_1,\ \alpha_1 = 3$.

Denote by $p_{4k}(M^n)$ the class of weak ∇-homologies corresponding to the class $P_{4k}(M^n)$. We shall be concerned with the reduced characteristic classes p_{4k}. Consider the $4k$-dimensional class

$$s_{4k}(M^n) = \varphi_k(p_4, \dots, p_{4k}).$$

Since $\beta_j \neq 0$ $(j = 1, \dots, k)$, the class p_{4k}, in turn, is determined by the classes $s_4, \dots, s_{4k}$, and in order to prove the topological invariance of the classes p_{4k} it is enough to prove the topological invariance of the classes s_{4k}. From formula (1) it follows that

$$s_{4k}(M^{4k}) = \alpha_k \Sigma^i_{4k}(M^{4k}), \tag{2}$$

so that for $n = 4k$ the class $s_{4k}(M^n)$ is topologically invariant.

The principal result of the present paper is that the class $s_{4k}(M^n)$ is topologically invariant also for $n = 4k + 1$. In particular, the reduced Pontryagin class $p_4(M^5)$ of a five-dimensional manifold M^5 is topologically invariant. This class, however, can no longer be expressed in terms of invariants of the ring of ∇-homologies of the manifold M^5.

2. Proof of the topological invariance of the class $s_{4k}(M^{4k+1})$. Let M^{4k+1} and M_1^{4k+1} be smooth compact orientable manifolds that coincide as topological manifolds. We must prove that

$$s_{4k}(M^{4k+1}) = s_{4k}(M_1^{4k+1}).$$

For this purpose we shall show that, whatever the $4k$-dimensional class u^{4k} of integral Δ-homologies of the manifold M^{4k+1}, the scalar products

$$(s_{4k}(M^{4k+1}), u^{4k}) \quad \text{and} \quad (s_{4k}(M_1^{4k+1}), u^{4k})$$

coincide.

Let V^{4k} be an oriented submanifold of the smooth manifold M^{4k+1} belonging to the class u^{4k}; let v^{4k} be its fundamental Δ-class; let i be the embedding of

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V^{4k} in M^{4k+1}, and i^*, i_* the corresponding homomorphisms of the groups of ∇- and Δ-homologies, so that $i_* v^{4k} = u^{4k}$. As is easily verified,

$$i^*[p_{4j}(M^{4k+1})] = p_{4j}(V^{4k}) \quad (j = 1, \dots, k).$$

Consequently,

$$i^*[s_{4k}(M^{4k+1})] = s_{4k}(V^{4k})$$

and

$$(s_{4k}(M^{4k+1}), u^{4k}) = (i^*[s_{4k}(M^{4k+1})], v^{4k}) = (s_{4k}(V^{4k}), v^{4k}), \tag{3}$$

i.e.

$$(s_{4k}(M^{4k+1}), u^{4k}) = \alpha_k \sigma(V^{4k})$$

(see (2)). In exactly the same way, if V_1^{4k} is an oriented submanifold of the smooth manifold M_1^{4k+1} belonging to the class u^{4k}, then

$$(s_{4k}(M_1^{4k+1}), u^{4k}) = \alpha_k \sigma(V_1^{4k}),$$

and it remains to prove the equality

$$\sigma(V^{4k}) = \sigma(V_1^{4k}).$$

Introduce a Riemannian metric in M^{4k+1} and construct a regular neighborhood of the submanifold V^{4k}, composed of pairwise nonintersecting geodesic normals. V^{4k} divides this neighborhood into two parts, and we choose V_1^{4k} inside one of them. Then V^{4k} and V_1^{4k} will bound, in the indicated neighborhood, an oriented (possibly not

smooth) manifold with boundary, whence it must be that $\sigma(V^{4k}) = \sigma(V^{4k})$ ([10]).

Remark. As is known, the definition of the characteristic ∇-classes of Pontryagin extends to nonorientable manifolds. Passing to the orientable double covering shows that the classes $s_{4k}(M^{4k+1})$ remain topologically invariant.

3. The class p_4 and the ring of ∇-homologies of a manifold.
We shall show that for $n > 4$ the class $p_4(M^n)$ is not determined by the ring of ∇-homologies of the manifold M^n.

Let Q^4 be the complex projective plane with homogeneous coordinates ξ, η, ζ, and let τ be the transformation taking the point $(\xi, \eta, \zeta) \in Q^4$ to the point $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ with complex-conjugate coordinates. Take the direct product $Q^4 \times [0, 1]$ of the manifold Q^4 with the interval $[0, 1]$, glue the boundary $Q^4 \times 0$ and $Q^4 \times 1$, identifying the points $q \times 0$ and $\tau(q) \times 1$ ($q \in Q^4$), and denote the resulting smooth compact orientable manifold by V^5. It was first considered by Wu ([4]). The groups of integral ∇-homologies of dimensions $0, 1, 4, 5$ of the manifold V^5 are free cyclic groups, the two-dimensional group is trivial, and the three-dimensional group has order 2. The multiplication table of the generators u_0, u_1, u_3, u_4, u_5 of the indicated groups can be represented in the form

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$$u_1 \smile u_4 = u_4 \smile u_1 = u_5, \qquad u_i \smile u_0 = u_0 \smile u_i = u_i \quad (i = 0, 1, 3, 4, 5);$$

the remaining products are equal to zero.

Put

$$A(u_1) = -u_1; \qquad A(u_4) = -u_4; \qquad A(u_i) = u_i \quad (i = 0, 3, 5).$$

Comparing these formulas with the multiplication table of the generators, we see that they define a certain automorphism A of the ring $H(V^5)$ of integral ∇-homologies of the manifold V^5. At the same time $P_4(V^5) = \pm 3u_4$, $A(P_4(V^5)) = -P_4(V^5) \neq P_4(V^5)$, so that $P_4(V^5)$ is not determined by the ring $H(V^5)$. Nor is $P_4(V^5)$ determined by the full ring of ∇-homologies, comprising all modular rings $H(V^5, m)$ $(m = 2, 3, ...)$, since the transformations induced by the automorphism A in these rings are also automorphisms. Since four-dimensional torsions are absent, the same is true also for the reduced class $\bar{p}_4(V^5)$.

To obtain a similar example for $n > 5$, it suffices to replace the manifold V^5 by its direct product with a torus of dimension $n - 5$.

4. Some properties of the classes $s_{4k}(M^{4k+1})$. Formula (3) makes it possible to carry over a number of theorems known for the classes $s_{4k}(M^{4k})$ to the classes $s_{4k}(M^{4k+1})$. For example, according to formula (2), the class $s_{4k}(M^{4k})$ is divisible by α_k. Comparing this fact with formula (3), we see that the class $s_{4k}(M^{4k+1})$ is also divisible by α_k. In particular, $p_4(M^5)$ is divisible by 3—a theorem which also follows from Wu' s results ([5]).

Here is a deeper theorem on five-dimensional manifolds. Denote by $\omega_2(M^n)$ the two-dimensional characteristic ∇-class of Stiefel—Whitney of the manifold M^n. According to the results of my paper ([6]), if $\omega_2(M^4) = 0$, then $p_4(M^4)$ is divisible by 48. From formula (3) it is not hard to derive that the same is true for five-dimensional manifolds: if $\omega_2(M^5) = 0$, then $p_4(M^5)$ is divisible by 48. Indeed, in the notation of § 2,

$$\omega_2(V^4) = i^*\omega_2(M^5) = 0;$$

therefore, $p_4(V^4)$ is divisible by 48 and, according to formula (3), $p_4(M^5)$ is also divisible by 48.

Ivanovo State
Pedagogical Institute

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Note: Figure translations are in progress. See original paper for figures.

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