S. V. NAGAEV

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Abstract

Full Text

S. V. NAGAEV

ON SOME LIMIT THEOREMS FOR HOMOGENEOUS MARKOV CHAINS

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Let an abstract space X be given, and let $\mathcal{F}_x$ be a σ-algebra of its subsets. Let $p(\eta, A)$, $\eta \in X$, $A \in \mathcal{F}_x$, be the transition-probability function. In what follows we shall assume that there exists a stationary probability distribution $p(A)$ such that, for some $\rho < 1$ and c,

$$|p^{(n)}(\eta, A) - p(A)| < c\rho^n \tag{1}$$

uniformly with respect to $\eta \in X$ and $A \in \mathcal{F}_x$, where $p^{(n)}(\eta, A)$ is the probability of transition in n steps from state η to a state belonging to the set A. The function $p(\eta, A)$, together with the initial probability distribution $\pi(A)$, determines a sequence $x_1, x_2, \dots, x_n, \dots$ of random variables connected in a homogeneous Markov chain, with

$$P(x_1 \in A) = \pi(A), \qquad P(x_n \in A) = \int_X p^{(n-1)}(\eta, A)\, \pi(d\eta). \tag{2}$$

Let $f(\eta)$ be a real function defined on X and measurable with respect to $\mathcal{F}_x$.

Theorem 1. If

$$\int |f(\eta)|^2\, p(d\eta) < \infty,$$

$$\sigma^2 = \lim_{n\to\infty} M \left[\frac{1}{\sqrt{n}} \sum_{m=1}^{n} (f(x_m) - Mf(x_m)) \right] > 0$$

(the mathematical expectation is computed under the assumption that the initial distribution is stationary), then for any initial distribution $\pi(A)$

$$\lim_{n\to\infty} P\left\{ \frac{1}{\sqrt{n}} \sum_{m=1}^{n} \left(f(x_m) - \int_X f(\eta)\, p(d\eta) \right) < x \right\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2\sigma^2}\, du. \tag{3}$$

This theorem is an analogue of the well-known theorem of P. Lévy, which states that if $x_1, x_2, \dots, x_n, \dots$ is a sequence of independent identically distributed random variables and $\sigma^2 = Dx_i < \infty$, then

$$\lim_{n\to\infty} P\left(\frac{\sum_{i=1}^n x_i - na}{\sigma\sqrt{n}} < x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du,$$

where a is the mathematical expectation of χ_i. Up to the present time the central limit theorem for a homogeneous Markov chain with an arbitrary set of states has been proved under the assumption that, for some $\delta > 0$ ([1], [2], [3]),

$$\int_X |f(\eta)|^{2+\delta} p(d\eta) < \infty.$$

Theorem 2. Let $u_1, u_2, \dots, u_n, \dots$ be a sequence of independent random variables with common distribution function $F(x)$ such that

$$F(x) = p(f(\eta) < x)$$

(here $p(A)$ is the stationary distribution). If, for some sequence of constants A_n and $B_n > 0$,

$$\lim_{n\to\infty} P\left(\frac{1}{B_n}\left(\sum_{m=1}^{n} u_m - A_n\right) < x\right) = V_\alpha(x),$$

where $V_\alpha(x)$ is a stable law with characteristic exponent α, and if for some $0 < \nu \le 1$

$$\lim_{n\to\infty} n^{1/2} B_n^{-\nu} \sup_{\xi} \int_{|f(\eta)|<B_n\tau} |f(\eta)|^{\nu} p(\xi, d\eta) = 0, \tag{4}$$

whatever $\tau > 0$ may be, then, for an arbitrary initial distribution $\pi(A)$,

$$\lim_{n\to\infty} P\left(\frac{1}{B_n}\left(\sum_{m=1}^{n} f(x_m) - A_n\right) < x\right) = V_\alpha(x). \tag{4'}$$

Condition (4) is satisfied, for example, when for some $\varepsilon < \alpha/2$ the moments

$$\int_X |f(\eta)|^{\alpha-\varepsilon} p(\xi, d\eta)$$

are uniformly bounded in ξ.

Theorem 3. If, for some $0 < \alpha < 2$,

$$\int_X |f(\eta)|^\alpha p(d\eta) < \infty,$$

then, for some choice of the constants A_n and for an arbitrary initial distribution,

$$\lim_{n\to\infty} P\left(\frac{1}{n^{1/\alpha}}\left(\sum_{m=1}^{n} f(x_m) - A_n\right) < x\right) = E(x), \tag{5}$$

where $E(x)$ is an improper law.

Corollary. If

$$\int_X |f(\eta)| p(d\eta) < \infty,$$

then the sequence $f(x_1), f(x_2), \ldots, f(x_n), \ldots$ obeys the law of large numbers.

Let now X be a countable set $\{\omega_i\}$ $(i = 1, 2, \ldots)$ and

$$\beta = \inf_{(i,j)} \sum_{k=1}^{\infty} \min(p_{ik}, p_{jk}), \tag{6}$$

where p_{ik} is the probability of transition from ω_i to ω_k in one step (the meaning of condition (6) is explained in ([3])).

Assume further that all states ω_i are essential and form a positive class ([4]). In view of (6), this class consists of a single subclass. Let $f(\omega_j) = a + k_j h$, where a is an arbitrary real number, k_j is an integer, and $h > 0$.

Theorem 4. If the greatest common divisor k_j is equal to 1,

$$\sum_{j=1}^{\infty} f^2(\omega_j) p_j < \infty$$

and $\sigma > 0$ (p_j are the final probabilities; σ is defined in the same way as in Theorem 1), then, uniformly with respect to s,

$$\lim_{n\to\infty}\left(\frac{\sigma\sqrt{n}}{h} P_{\pi n}(s) - \frac{1}{\sqrt{2\pi}} e^{-{z_{ns}}^{2/2}}\right) = 0, \tag{7}$$

where $P_{\pi n}$ is the probability that

$$\sum_{m=1}^{n} f(x_m) = an + sh,$$

under the condition that the initial distribution is $\pi(A)$, and

$$z_{ns} = \frac{1}{\sigma\sqrt{n}}\left(an + sh - n\sum_{j=1}^{\infty} f(\omega_j)p_j\right).$$

Theorem 5. If the conditions of Theorem 4 are satisfied and, in addition, for some integer $k \geq 3$ and some $\delta > 0$

$$\sum_{j=1}^{\infty} |f(\omega_j)|^{k+\delta} p_{ij} < M < \infty$$

uniformly for all i, then

$$P_{\pi n}(s) = \frac{h}{\sigma\sqrt{n}}\left\{\varphi(z_{ns}) + \sum_{m=1}^{k-2}\frac{1}{n^{m/2}}T_{\pi m}(\varphi(z_{ns})) + O\left(\frac{1}{n^{(k-2)/2}}\right)\right\}. \tag{8}$$

Here

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \qquad T_{\pi m}(\varphi(x)) = \sum_{i=m+2}^{3m} a_{\pi m i}\frac{d^i}{dx^i}\varphi(x),$$

where the coefficients $a_{\pi m i}$ depend on the initial distribution $\pi(A)$ and on $p(\eta, A)$.

Theorems 4 and 5 are a generalization of results of S. Kh. Sirazhdinov (5). All the theorems formulated above are obtained by means of a method based on the application of the spectral theory of linear operators in a Banach space (7). This method is a natural generalization of the matrix method and the method of integral equations, presented, for example, in (6); it can also be used for proving multidimensional limit theorems.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.

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