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Abstract

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ON THE QUESTION OF FLUCTUATIONS OF THE PARAMETERS OF CERTAIN LINEAR SYSTEMS

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1. A specific feature of the problem of the scattering of waves propagating in a shielded transmission line by random inhomogeneities present in the line is the circumstance that the secondary waves (reradiated by the inhomogeneities) are guided by the same path as the primary wave. If the length of the line is sufficiently great, then the secondary field (i.e., the amplitudes of the reflected wave and of waves of other types arising as a result of retransformation) may turn out to be comparable with the field of the incident wave. For this reason the perturbation method, usually used in scattering problems [(1,2)] and not taking account of the secondary reradiation of the scattered waves, proves insufficient for solving a number of questions concerning the influence of random inhomogeneities on the characteristics of transmission lines.

Analogous difficulties also arise in solving a number of other problems in which the question concerns the influence of random deviations of parameters on the characteristics of a linear system. As examples one may cite: a filter (or delay line) the parameters of whose cells have a certain random scatter about the nominal values; a traveling-wave tube in which a slow-wave system with random structural defects is used; an oscillatory circuit whose parameters change by jumps of a random quantity, etc. The solution of such problems can be reduced, under certain assumptions (see, for example, [(3)]), to the investigation of a system of linear difference equations

$$Y_j(n) = A_{jk}(n)Y_k(n-1), \qquad j,k = 1,2,\dots,L, \tag{1}$$

whose coefficients are random functions of n.

The determination of the most complete characteristic of the random process $Y_j(n)$—the probability-density distribution $W(Y_j, n)$—is associated with considerable difficulties and can be carried out either by investigating the general solution of system (1), or by solving the corresponding differential-difference equations for W. In many cases, however, the necessary information about the random process is given by moments and correlation functions.

In the present work the general solution of system (1) is given and a comparatively simple method is presented for calculating moments and correlation functions. As an example, a chain of the simplest Γ-shaped four-terminal networks and an oscillatory circuit with fluctuating parameters are considered.

2. To find the solution of a system of linear difference equations with variable coefficients, the method of successive approximations may be used. In contrast to differential equations, where in each individual case it is necessary to prove the convergence of the series of approximations, here the zero approximation can always be chosen in such a way that, on any bounded interval, a finite number of successive approximations leads to the exact solution. In this case, obviously—

but the initial assumption of smallness of the perturbation of the coefficients becomes superfluous.

Let us write the coefficients of equations (1) in the form

$$A_{jk}(n) = A^0_{jk} + \mu a_{jk}(n). \tag{2}$$

The solution of system (1) satisfying the initial conditions

$$Y_j(n)\big|_{n=0} = C_j, \tag{3}$$

will be sought in the form of a series in powers of μ

$$Y_j(n) = \sum_{s=0}^{\infty} \mu^s Y_j^{(s)}(n). \tag{4}$$

Then, substituting (4) into (1) and grouping terms with equal powers of μ, we obtain

$$Y_j^s(n) = A^0_{jk} Y_k^{(s)}(n-1) + a_{jk}(n) Y_k^{(s-1)}(n-1). \tag{5}$$

If we require that $Y_j^{(0)}(n)$ satisfy the initial conditions (3), then, as is not difficult to see, the series terminates at $s = n$, since all $Y_j^{(s)}$ for $s > n$ vanish identically. Indeed, $Y^{(s)}(0) \equiv 0$ for $s > 0$ by the choice of the zeroth approximation. From (5) it is obvious that $Y_j^{(s)}(n) = 0$, if $Y_j^{(s-1)}(n-1) = 0$ and $Y^{(s)}(n-1) = 0$. According to the method of induction it follows from this that $Y^{(s)}(n) \equiv 0$ for $s > n$. By choosing C_j in the appropriate way, one can satisfy any boundary conditions (not necessarily specified at $n = 0$).

For example, the general solution of the second-order equation with one variable coefficient*

$$Y(n+1) - \{\Phi_0 + P(n)\}Y(n) + Y(n-1) = 0, \tag{6}$$

obtained by the method of successive approximations, has the form

$$\begin{aligned} Y(n) = {} & Ae^{j\varphi_0 n}\times \\ & \left\{1 + \sum_{s=1}^{n-1}(2j\sin\varphi_0)^{-s}\sum_{f_s=1}^{n-1}\sum_{f_{s-1}=1}^{f_s-1}\cdots\sum_{f_1=1}^{f_2-1}\prod_{m=1}^{s}P_{f_m}\left[e^{j2\varphi_0(f_m-f_{m+1})}-1\right]\right\} \\ & + Be^{-j\varphi_0 n}\times \\ & \left\{1 + \sum_{s=1}^{n-1}(2j\sin\varphi_0)^{-s}\sum_{f_s=1}^{n-1}\sum_{f_{s-1}=1}^{f_s-1}\cdots\sum_{f_1=1}^{f_2-1}\prod_{m=1}^{s}P_{f_m}\left[1-e^{-j2\varphi_0(f_m-f_{m+1})}\right]\right\}, \end{aligned} \tag{7}$$

where A and B are arbitrary constants and $\varphi_0 = \arccos(\Phi_0/2)$.

If the quantities $P(n)$ are relatively small ($\mu \ll 1$), then in many cases one may restrict oneself to the first (3) or second approximation.

Increasing the order of the difference equation (system) and complicating the coefficients will lead to a further increase in the cumbersomeness of the solution, which limits the possibility of its application in the study of random processes. At the same time, if the number of cells is small (for example, in filters), a general solution of type (7) can be successfully used for

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* An equation of this type is obtained in the investigation of a chain of four-terminal networks either with variable series resistance or with variable shunt capacitance.

investigation of perturbations caused by random changes in the parameters of the cells.

3. The mean* characteristics of the quantities described by (1) can be found rather simply in the case when the process $Y_j(n)$ is a simple Markov chain. The latter occurs if:

a) the random functions $A_{jk}(n)$ are uncorrelated**, i.e.

$$\overline{A_{jk}(n)A_{lp}(m)} = \overline{A_{jk}(n)}\,\overline{A_{lp}(m)}; \tag{8}$$

b) the boundary (initial) conditions are specified for one and the same value of n, for example for $n = 0$,

$$\left.Y_j(n)\right|_{n=0} = Y_j^0. \tag{9}$$

In this case, the equations and initial conditions for $\overline{Y_j(n)}$ are obtained immediately by averaging (1) and (9), i.e.

$$\overline{Y_j(n)} = \overline{A_{jk}(n)}\,\overline{Y_k(n-1)}; \qquad \left.\overline{Y_j(n)}\right|_{n=0} = Y_j^0. \tag{10}$$

Thus, the mean values $\overline{Y_j(n)}$ in chains with random parameters, when conditions (8) and (9) are fulfilled, are distributed in the same way as the quantities $Y_j(n)$ in a chain with mean parameter values $\overline{A_{jk}(n)}$ and averaged initial conditions***.

Under the same assumptions (8) and (9), the system of equations for the mean values of the products $\overline{Y_j(n)Y_k^*(n)} = \xi_{jk}(n,0)$ is obtained after averaging the products of the equations (respectively, the right- and left-hand sides) of system (1) by the equations of the complex-conjugate system****. As a result we obtain L^2 equations

$$\xi_{jk}(n,0) = \overline{A_{jl}(n)A_{kp}^*(n)}\,\xi_{lp}(n-1,0). \tag{11}$$

If the elements of the coefficient matrix (11) do not depend on n, then, putting***** $\xi_{jk}(n,0) = \xi_{jk}(0,0)\alpha^n$, from the condition for the existence of a nontrivial solution we find

$$\det\left|\overline{A_{jl}(n)A_{kp}^*(n)} - \alpha\delta_{jl}\delta_{kp}\right| = 0, \tag{12}$$

where δ_{qf} is the Kronecker symbol. Equation (12) determines L^2 values of α and, consequently, L^2 linearly independent solutions $\xi_{jk}(n,0)$, with the aid of which the initial conditions can be satisfied

$$\left.\xi_{jk}(n,0)\right|_{n=0} = \overline{Y_j^0 Y_k^{0*}}. \tag{13}$$

Multiplying (1) by $Y_k^*(n-m)$ and averaging, for the correlation functions

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* Here and below, averaging over the ensemble is meant.

** In the presence of correlation in a finite region, i.e. if (8) is satisfied only for $|m-n| \geq \nu \neq 0$, the process $Y_j(n)$ is a complex Markov chain. The latter can be reduced to a simple chain (4), but the equations become considerably more complicated.

*** Let us note that, generally speaking, the mean value of the voltage and current in an ensemble of chains of lossless quadripoles may increase or decrease according to an exponential law.

**** Equations for second-order moments (or other mean quantities) can also be obtained in other ways. In particular, one may use the differential-difference equation (which here will be of infinite order) for the transition probabilities, or pass from the difference equation (1) to the summation equation (the analogue of an integral equation) and use the method applied in [5] for the study of an integral equation with a random kernel. The method used here leads to the required result by a shorter route.

***** If the statistical properties of A_{jk} depend on n (for example, random perturbations are superimposed on a prescribed dependence $A_{jk}(n)$), then $\overline{A_{jl}(n)A^*_{kp}(n)} \neq$ const, and the solution of system (11) is written in the form of a finite number of multiple sums (Sec. 2). The difficulty of investigating the solution in each particular case will be determined by the character of the dependence of A_{jk} on n.

$\xi_{jk}(n,m) = \overline{Y_j(n)Y^*_k(n-m)}$, we obtain L systems (corresponding to the index k, which assumes values from 1 to L), each of which consists of L difference equations

$$\xi_{jk}(n,m) = A_{jl}(n)\xi_{lk}(n-1,m-1). \tag{14}$$

The solution of the systems (14), when the $\xi_{jk}(n,0)$ are known, is found easily. In an analogous way, moments of higher orders can also be calculated.

4. The distribution of the mean square of the modulus of the voltage in a chain of the simplest Γ-shaped four-terminal networks with fluctuating series resistances $(X_n = X_0\{1+P(n)\})$, obtained by the method presented, under the assumption that $\overline{P(n)P(m)} = p^2\delta_{nm}$, $p^2 \ll 1$, and that the load at the output is matched, has the form:

$$\overline{|V(n)|^2} \simeq |V_0^2| \left\{ \left(1 - p^2 \operatorname{tg}^2 \frac{\varphi}{2}\right) \exp\left(2p^2 n \operatorname{tg}^2 \frac{\varphi}{2}\right) - \right.$$

$$\left. - p^2 \operatorname{tg}^2 \frac{\varphi}{2} \frac{\sin(2n-1)\varphi}{\sin\varphi} \cdot \exp\left(-p^2 n \operatorname{tg}^2 \frac{\varphi}{2}\right) \right\}, \tag{15}$$

where φ is the phase shift per four-terminal network. It follows from (15) that for large n the quantity $\overline{|V^2(n)|}$ increases according to an exponential law. Similarly, the mean square of the modulus of the current in an oscillatory circuit with a capacitance fluctuating according to the law $C(t) = C_0\{1+P(n)\}^{-2}$, where t is time and n is the integer part of t/τ, is described by the formula

$$\overline{|I^2(n)|} \simeq I_0^2 \left\{ \left[1 - p^2\left(\sin^2\omega\tau - \frac{\omega\tau\sin 2\omega\tau}{2\sin^2\omega\tau}\right)\right] \exp(2p^2 n \sin^2\omega\tau) - \right.$$

$$-p^2 \left[\frac{\sin^2 \omega\tau \cdot \sin 2(n-1)\omega\tau}{\sin 2\omega\tau} + \frac{\omega\tau \cos(2n+1)\omega\tau}{\sin \omega\tau} \right] \exp(-p^2 n \sin^2 \omega\tau) \Bigg\}, \quad (16)$$

where ω is the resonant frequency of the ideal circuit; τ is the time after which the capacitance changes.

Equation (16) shows that, because of capacitance fluctuations, the energy stored in an ensemble of circuits increases with time; moreover, as $t \to \infty$, maxima of $\frac{1}{2}L\left[\overline{I^2(t,\tau)}\right]_{t=\mathrm{const}}$ are observed at values of τ satisfying the equation $2\omega\tau \operatorname{ctg} \omega\tau = 1$. The latter coincides with the condition for a maximum of the mean square of the density of the amplitude spectrum of the function $C(t)$ at one of the frequencies corresponding to parametric resonance (6). The results obtained are not difficult to generalize to systems with randomly varying intervals τ.

5. The method presented for calculating averaged quantities can be used not only in the study of systems with stepwise variation of parameters, but also of systems with continuous fluctuations, if the form of the latter (in space or time) may, to a certain degree of accuracy, be regarded as repeating. In this way, for example, one can estimate the influence of inhomogeneities in cylindrical transmission lines [7].

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