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T. Ya. Zagorskii

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Abstract

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T. Ya. Zagorskii

SOME MIXED PROBLEMS FOR PARABOLIC SYSTEMS OF DIFFERENTIAL EQUATIONS

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In the present note a mixed problem for a parabolic system is reduced to solvable integral equations. As was done for elliptic systems by Z. Ya. Shapiro (^1) and Ya. B. Lopatinskii (^2), the kernel used here is the Green function, constructed by us earlier (^3), of the mixed problem for a half-space.

I. In the paper the following notation is used:

$$A\left(\frac{\partial}{\partial x}\right) = \left\| \sum_{k_1+\cdots+k_n=s} a_{ij}^{(k_1,\ldots,k_n)} \frac{\partial^s}{\partial x_1 \cdots \partial x_n} \right\|;$$

$$B_1\left(\frac{\partial}{\partial x}\right) = \left\| \sum_{k_1+\cdots+k_n=s_1} b_{rl}^{(k_1,\ldots,k_n)} \frac{\partial^{s_1}}{\partial x_1 \cdots \partial x_n} \right\|$$

$$(i = 1, 2, \ldots, N; \quad j = 1, 2, \ldots, N; \quad r = 1, 2, \ldots, sN/2; \quad l = 1, 2, \ldots, N);$$

$a_{ij}^{(k_1,\ldots,k_n)}$, $b_{rl}^{(k_1,\ldots,k_n)}$ are certain constant numbers; s is a positive even number; s_1 is a positive integer, $s_1 < s$; $x = (x_1, \ldots, x_n)$ is a point in n-dimensional space; $x' = (x_1, \ldots, x_{n-1})$.

The problem under consideration can be formulated as follows: it is required to determine a solution of the system

$$\frac{\partial u}{\partial t} = A\left(\frac{\partial}{\partial x}\right) u \tag{1}$$

in a domain v, bounded by a convex surface S of Lyapunov type, satisfying the conditions

$$\text{for } t = 0 \qquad u = 0; \tag{2}$$

$$\lim_{x\to y\in S} B_1\left(\frac{\partial}{\partial x}\right) u = f_1(y;t); \tag{2'}$$

here u is a vector with components $u_1, \dots, u_N$; $f(y;t)$ is a continuous vector with components $f_1(y;t), \dots, f_{sN/2}(y;t)$, prescribed at each point $y \in S$ for $t > 0$;

$$|f_i(y,t)| < ce^{\sigma t}; \qquad c > 0; \qquad \sigma > 0. \tag{α}$$

The system (1) is parabolic, and for every half-space bounded by the plane tangent to S at a point $y \in S$, with ort-normal ν_y directed into the volume v, the solvability condition (^3) is satisfied.

II. In the paper (^3) we constructed a solution of system (1) in the half-space $x_n > 0$ with the initial condition (2) and with the boundary condition $\lim_{x_n\to 0} B(\partial/\partial x)u(x;t) = f(x';t)$. In contrast to the matrix $B_1(\partial/\partial x)$, in the matrix $B(\partial/\partial x)$ the order of differentiation in the i-th row is equal to $s_i < s$, so that $B(\partial/\partial x) = B_1(\partial/\partial x)$ if one sets $s_i = s_1$; in addition to the conditions (α), the function $f(x';t)$ is subject to one further condition at infinity. This

the solution has the form

$$u = \frac{1}{(2\pi)^{(n-1)/2}} \int_0^t d\tau \int_{-\infty}^{\infty} G(x' - \xi';\, x_n;\, t - \tau) f(\xi';\tau) d\xi'.$$

The derivative of order l of the Green function G with respect to some sequence of the arguments $x_1, \dots, x_n$, for the boundary matrix $B_1(\partial/\partial x)$, has the estimate

$$|D^{lG}| \le \frac{c\exp\{-c_1|x' - \xi'|^{s/(s-1)}(t-\tau)^{-1/(s-1)}\}}{(t-\tau)^{(s+n+l-s_1-1)/s}\{1 + x_n(t-\tau)^{-1/s}\}^{s+n+l-s_1-1}}. \tag{3}$$

The solution of the same problem for the half-space bounded by the plane P with orthonormal ν, situated arbitrarily with respect to the coordinate system, is written in the form

$$u = \frac{1}{(2\pi)^{(n-1)/2}} \int_0^t d\tau \int_{(P)} G^{\nu}(x_p - y;\, x - x_p;\, t - \tau) f(y;\tau) d^*_{yP};$$

here x_p is the projection of the point $x \in v$ onto the plane P, $y \in P$; inequality (3) for the function G^ν has the form

$$|D^{lG^\nu}| \le \frac{c\exp\{-c_1|x_p - y|^{s/(s-1)}(t-\tau)^{-1/(s-1)}\}}{(t-\tau)^{(s+n+l-s_1-1)/s}[1 + |x - x_p|(t-\tau)^{-1/(s-1)}]^{s+n+l-s_1-1}}. \tag{4}$$

Near the bounding plane P, for B_1G^ν one can obtain an estimate of the form

$$|B_1G^\nu| \leq \frac{c\exp\{-c_1|x_p - y|^{s/(s-1)}(t-\tau)^{-1/(s-1)}\}\,|x - x_p|}{(t-\tau)^{(s+n)/s}}. \tag{5}$$

The idea of the proof of this inequality is as follows. Let $f(\xi';\tau) = 0$ for $|x' - \xi'| < \varepsilon$ and $(t-\tau) < \varepsilon$. Applying the boundary condition, we obtain

$$\lim_{x_n \to 0} B_1\left(\frac{\partial}{\partial x}\right)\int_0^t d\tau \int_{-\infty}^{\infty} G(x' - \xi';\, x_n;\, t-\tau) f(\xi';\tau)d\xi' = 0.$$

Passing to the limit and differentiating under the integral sign (which is legitimate here), and applying the fundamental lemma of the calculus of variations, we conclude that

$$B_1(\partial/\partial x)G(x' - \xi';\, x_n;\, t-\tau)\big|_{x_n=+0} = 0$$

for all ξ' and τ, except for some neighborhood of the point $(x';t)$; observing that

$$B_1G = \int_0^{x_n} \frac{\partial}{\partial x_n}(B_1G)\,dx_n$$

and using estimate (3), we obtain estimate (5).

III. The solution of system (1) under conditions (2) and (2′) is sought in the form

$$u = \frac{1}{(2\pi)^{(n-1)/2}}\int_0^t d\tau \int_{(S)} G^{\nu_y}(x-y;\, t-\tau)\varphi(y;\tau)d_{yS};$$

ν_y is the orthonormal to the plane tangent to the surface S at the point $y \in S$, directed into the volume v. The integration is carried out over the surface S with variable point y; $\varphi(y;t)$ is a certain continuous vector-function. The vector-function $u(x;t)$ so defined satisfies system (1) and the initial condition (2); it remains to choose $\varphi(y;t)$ so that the boundary condition (2′) is satisfied.

The question is reduced to the solution of an integral equation on the basis of the following lemma on the jump of the integral $B_1\left(\frac{\partial}{\partial x}\right)u_1(x;t)$.

* In what follows, instead of $G^{\nu}(x_p - y;\ x - x_p;\ t - \tau)$ we shall write $G^{\nu}(x - y;\ t - \tau)$.

Lemma 1.

$$\lim_{x \to z \in S} B_1\left(\frac{\partial}{\partial x}\right) u(x;t) = B_1\left(\frac{\partial}{\partial x}\right) u(x;t)\Big|_{x=z} + \varphi(z;t).$$

For the proof, the surface S is divided into two parts: a small part S_1, containing the point z, and the remaining part S_2. Now one may write

$$B_1\left(\frac{\partial}{\partial x}\right) u(x;t) = I_1(x;t) + I_2(x;t).$$

Here

$$I_i(x;t) = \frac{1}{(2\pi)^{(n-1)/2}} \int_0^t d\tau \int_{(S_i)} B_1\left(\frac{\partial}{\partial x}\right) G^{\nu_y}(x-y; t-\tau)\varphi(y;\tau)\, d_{yS} \quad (i = 1, 2).$$

Since in the integral I_2 the point $y \neq z$, passage to the limit and differentiation under the integral sign are legitimate here, and consequently

$$\lim_{x \to z \in S} I_2(x;t) = I_2(z;t).$$

It remains to show that, for sufficiently small diameter of the portion S_1 and for $|x - z|$, the inequality

$$|I_1(x;t) - \varphi(z;t)| < \varepsilon$$

holds.

By virtue of the smallness of the portion S_1 and the convexity of S, one may assume that S_1 is projected one-to-one onto the plane T_z tangent to S at the point z. Now

$$I_1(x;t) = \frac{1}{(2\pi)^{(n-1)/2}} \int_0^t d\tau \int_{(T'_z)} B_1\left(\frac{\partial}{\partial x}\right) G^{\nu_z}(x-y; t-\tau)\varphi(y;\tau) \frac{d_{yT}z}{\cos(\nu_y \nu_z)} +$$

$$+ \frac{1}{(2\pi)^{(n-1)/2}} \int_0^t d\tau \int_{(S_1)} \left\{ B_1\left(\frac{\partial}{\partial x}\right) G^{\nu_y}(x-y;\ t-\tau) - B_1\left(\frac{\partial}{\partial x}\right) G^{\nu_z}(x-y;\ t-\tau) \right\} \varphi(y;\tau)\, d_{yS};$$

(T'_z) is the projection of the region S_1 onto the plane T_z.

From the results of the work (3) it follows that the first of these integrals differs little from $\varphi(z;t)$ ($\cos(\nu_y \nu_z)$ is close to unity). It is proved that the second integral tends to zero together with the diameter of the region S_1. Using this lemma and the boundary condition (2′), we obtain an integral equation for determining $\varphi(z,t)$. It has the form

$$\varphi(z;t) + \frac{1}{(2\pi)^{(n-1)/2}} \int_0^t d\tau \int_{(S)} B_1\left(\frac{\partial}{\partial x}\right) G^{\nu_y}(x-y;\ t-\tau)\Big|_{x=z} \varphi(y;\tau)\, d_{yS} =$$

$$= f(z;\ t). \qquad (6)$$

IV. For solving equation (6) we apply the method of successive approximations. Put

$$\varphi_0(z;t) = f(z;t)$$

and determine $\varphi_n(z;t)$ from the recurrence relation

$$\varphi_n(z;t) = -\frac{1}{(2\pi)^{(n-1)/2}} \int_0^t d\tau \int_{(S)} B_1\left(\frac{\partial}{\partial x}\right) G^{\nu_y}(x-y;t-\tau)\Big|_{x=z} \varphi(y;\tau)\, d_{yS}.$$

The infinite series $\varphi_0 + \varphi_1 + \cdots + \varphi_n + \cdots$ represents a formal solution of equation (6). Let

$$|\varphi_{n-1}(z;t)| < k\psi(t) \quad (k > 0).$$

Applying estimate (5), we obtain for $\varphi_n(z;t)$

$$|\varphi_n(z;t)| \le kc \int_0^t \psi(\tau)\, d\tau \int_{(S)} \frac{\exp\left[-c_1 |z_{T_y} - y|^{s/(s-1)} (t-\tau)^{-1/(s-1)}\right] |z - z_{T_y}|}{(t-\tau)^{(s+n)/s}}\, d_y S.$$

Let α be the angle between the vector $z - y$ and ν_y, and let $r = |z-y|$. We shall assume that $0 < \alpha_0 \le \alpha \le \pi/2$. Then

$$|z_{T_y} - y| = r \sin\alpha; \qquad |z - z_{T_y}| = r\cos\alpha;$$

$$|\varphi_n(z;t)| \le$$

$$\leq kc\int_0^t \frac{\psi(\tau)\,d\tau}{(t-\tau)^{(s+n)/s}}\int_S \exp\left[-c_1 r^{s/(s-1)}\sin^{s/(s-1)}\alpha\,(t-\alpha)^{-1/(s-1)}\right] r\cos\alpha\,d_yS.$$

From the Lyapunov condition it follows that $\cos\alpha \leq \lambda r^{\mu}$ $(\lambda > 0;\ 0 < \mu \leq 1)$, and therefore

$$|\varphi_n(z;t)| \leq$$

$$\leq kc\int_0^t \frac{\psi(\tau)\,d\tau}{(t-\tau)^{(n+s)/s}}\int_S \exp\left[-c_2 r^{s/(s-1)}(t-\tau)^{-1/(s-1)}\right] r^{1+\mu}\,d_yS \qquad (c_2 > 0). \tag{7}$$

From the point y, as from a center, draw a sphere of radius r, which cuts out on the surface S a portion of measure $\omega(r)$. Since this portion is an $(n-1)$-dimensional volume, two positive numbers a and b can be found such that the inequalities

$$ar^{n-1} \leq \omega(r) \leq br^{n-1}$$

hold.

Strengthening inequality (7), we write

$$|\varphi_n(z;t)| \leq c_3\int_0^t \frac{\psi(\tau)\,d\tau}{(t-\tau)^{(n+s)/s}}\times$$

$$\times\int_0^\infty \exp\left[-c_2\omega^{s/((s-1)(n-1))}(t-\tau)^{-1/(s-1)}\right]\omega^{(1+\mu)/(n-1)}\,d\omega \qquad (c_3 > 0).$$

Putting $\omega^{1/(n-1)} = \theta(t-\tau)^{1/s}$, again strengthening the inequality and changing the values of the parameters, we write

$$|\varphi_n(z;t)| \leq$$

$$\leq c_3\int_0^t \frac{\psi(\tau)\,d\tau}{(t-\tau)^{(s-\mu)/s}}\int_0^\infty \exp\left[-c_2\theta^{s/(s-1)}\right]\theta^{\mu+n-1}\,d\theta \leq c_4\int_0^t \frac{\psi(\tau)\,d\tau}{(t-\tau)^{(s-\mu)/s}} \qquad (c_4 > 0).$$

Using the estimate and condition (α), we obtain

$$|\varphi_n(z;t)| \leq c_4 \frac{\mu}{s} e^{\sigma t} \frac{\Gamma^n(\mu/s)}{n\Gamma(n\mu/s)}\, t^{n\mu/s},$$

which ensures the absolute and uniform convergence of $\sum_{n=0}^{\infty} \varphi_n(z;t)$.

Imposing additional restrictions on the function $f(y;t)$ ($f(y;0) = 0$), one can extend the results obtained to the case of more general boundary conditions characterized by the matrix $B(\partial/\partial x)$.

The generalization of the results of this article to the case of variable coefficients of the matrix of the boundary conditions, continuously dependent on $y \in S$, is achieved without imposing additional conditions.

Lviv Polytechnic Institute

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Note: Figure translations are in progress. See original paper for figures.