ON FUNCTIONS OF THREE VARIABLES

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Abstract

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Below we briefly indicate a method of proof of a theorem which gives a complete solution of Hilbert's 13th problem (in the sense of refuting the hypothesis stated by Hilbert).

Theorem 1. *Every prescribed real continuous function $f(x_1, x_2, x_3)$ of three variables on the unit cube E^3 can be represented in the form*

$$f(x_1, x_2, x_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} h_{ij} \left[\varphi_{ij}(x_1, x_2), x_3\right], \tag{1}$$

where the functions of two variables h_{ij} and φ_{ij} are real and continuous.

A. N. Kolmogorov recently obtained ([1]) the representation

$$f(x_1, x_2, x_3) = \sum_{i=1}^{3} h_i \left[\varphi_i(x_1, x_2), x_3\right], \tag{2}$$

where the functions h_i and φ_i are continuous, the functions h_i are real, and the functions φ_i take values belonging to a certain tree Ξ. The tree Ξ in A. N. Kolmogorov' s construction (for the case of a function of three variables) may be taken not universal, but such that all its points have branching index not exceeding 3. For this purpose the functions u^r_{km} of the basic lemma ([1]) (for $n = 2$) should be chosen so that, in addition to the five properties indicated there, they also have the following properties:

6) *The boundary of each level set of each function u^r_{km} divides the plane into no more than 3 parts.*
7) *For every r, $G^r_{11} \supset E^2$.*

By virtue of this remark, Theorem 1 is a consequence of the existence of the representation (2) and of the following theorem:

Theorem 2. *Whatever the family F of real equicontinuous functions $f(\xi)$, defined on a tree Ξ all of whose points have branching index ≤ 3, one can realize the tree as a subset X of the three-dimensional cube E^3 in such a way that any function of the family F can be represented in the form*

$$f(\xi) = \sum_{k=1}^{3} f_k(x_k),$$

where $x = (x_1, x_2, x_3)$ is the image of $\xi \in \Xi$ in the tree X; $f_k(x_k)$ are continuous real functions of one variable, with f_k depending continuously on f (in the sense of uniform convergence).

Let us introduce some auxiliary notions. Let K be a finite segmental complex situated in E^3 and consisting of segments not parallel to any of the coordinate planes.

Definition 1. A system of points of K

$$a_0 \neq a_1 \neq \cdots \neq a_{n-1} \neq a_n$$

is called a **lightning**, if the segments $\overline{a_{i-1}a_i}$ are perpendicular, respectively, to the axes X_{α_i} and

$$\alpha_1 \neq \alpha_2 \neq \cdots \neq \alpha_{n-1} \neq \alpha_n.$$

A finite system of pairwise distinct points $a_{i_1 i_2 \dots i_n}$, numbered by tuples of indices $i_1 i_2 \dots i_n$, is called a **branching scheme** if: 1) there exists only one point a_0, numbered by a single index; 2) together with $a_{i_1 i_2 \dots i_{n-1} i_n}$, the system contains $a_{i_1 \dots i_{n-1}}$.

Definition 2. A branching system of points $a_{i_1 \dots i_n}$ situated on K is called a **deriving scheme** if, for a fixed tuple $i_1 \dots i_n$, the totality of points of the form $a_{i_1 \dots i_n i_{n+1}}$ lies in the plane passing through $a_{i_1 \dots i_n}$, perpendicular to some coordinate axis $x_{\alpha_{i_1 \dots i_n}}$, and exhausts all points of intersection of this plane with K distinct from $a_{i_1 \dots i_n}$.

The tree Ξ can be represented in the form

$$\Xi = \overline{\bigcup_{n=1}^{\infty} D_n}, \qquad D_n \subset D_{n+1},$$

where D_n are finite trees, D_1 is a simple arc, and D_{n+1} is obtained from D_n by attaching, at some point p_n, which is for D_n neither a branching point nor an endpoint, the segment S_n ([2]).

Denote by ω_n the upper bound of the oscillations of functions $f \in F$ on the components of the difference $\Xi \setminus D_n$. It is easy to see that

$$\omega_n \to 0 \quad \text{as } n \to \infty.$$

Therefore one can choose a sequence

$$n_1 < n_2 < \dots < n_r < \dots,$$

such that

$$\omega_n \leq \frac{1}{r^2} \quad \text{for } n \geq n_r.$$

A realization X of the tree Ξ in E^3 is constructed in the form:

$$X = \overline{\bigcup_{n=1}^{\infty} D'_n},$$

where D'_n are segmental complexes realizing D_n in such a way that the images S'_n of the arcs S_n are segments not perpendicular to the coordinate axes.

The inductive construction of D'_n is carried out so that $\overline{\bigcup_{n=1}^{\infty} D'_n}$ is a tree ([2]) and with the following conditions satisfied:

1) Every function $f \in F$ is represented on D_n in the form

$$f(\xi) = \sum_{k=1}^{3} f_k^n(x_k), \tag{3}$$

where $f_k^n(x_k)$ depends continuously on f.

2) The tree D'_n from any point a_0 has an outgoing scheme in which the first direction α_0 may be chosen arbitrarily.

3) Let A_n be the set of points of D'_n that are images of branching points of E. There exists a countable set $B_n \subset D'_n$, $B_n \cap A_n = 0$, such that broken lines $a_0 \dots a_m$, beginning at $a_0 \in D'_n \setminus B_n$, have no common points with A_n and no coincident points $a_i = a_j$, $i \neq j$.

4) If $n_r < n \leqslant n_{r+1}$, then

$$|f_k^n(x_k) - f_k^{n_r}(x_k)| \leqslant \left(3 + \frac{n - n_r}{n_{r+1} - n_r}\right) \frac{1}{r^2}. \tag{4}$$

The proof of the possibility of the inductive construction of the trees D'_n and the functions f_k^n, while preserving properties 1)–4), is too complicated to be presented here. Roughly speaking, at each step the attached segment S'_{n+1} is chosen very short; its direction and the manner of mapping S_{n+1} onto S'_{n+1} are chosen so as to ensure the fulfillment of properties 2) and 3) for D'_{n+1}. Preservation of equality (3) in the transition from n to $n+1$ on the newly attached segment S_{n+1} requires introducing a correction $f_k^{n+1} - f_k^n$ to at least one of the functions f_k^n on the projection of S'_{n+1} onto the axis x_k. In order to preserve equality (3) on the previously constructed tree D'_n, this correction must be compensated by new corrections to the functions f_k^n on a number of other segments. We do not set out here the exact method of introducing these corrections. We note only the following: these corrections must be such that, for $n' = n + 1$, inequality (4) is preserved; if S'_{n+1} is sufficiently small and suitably placed, they can be made, for each function f_k^n, on a finite system of pairwise nonintersecting segments of the axis x_k; in the proof of this possibility, the circumstance that the tree D'_n has properties 2) and 3) is used essentially.

The proof of the existence of a continuous function

$$f_k(x_k) = \lim_{n\to\infty} f_k^n(x_k)$$

and of the validity of the equality

$$f(\xi) = \sum_{k=1}^{3} f_k(x_k)$$

on all of X is not difficult.

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References

[1] A. N. Kolmogorov, DAN, 108, No. 2, 179 (1956).
[2] K. Menger, *Kurventheorie*, X, Berlin–Leipzig, 1932.

Note: Figure translations are in progress. See original paper for figures.