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Abstract

Full Text

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ON THE FACTORIZATION OF Π-SEPARABLE GROUPS

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In the work of S. A. Chunikhin (1) the following theorem is proved.
Let $m > 1$ be the greatest Π-Sylow divisor of the order g of a finite Π-separable group $\mathfrak{G}$, and let

$$m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

where $p_1, p_2, \dots, p_k$ are all the distinct prime divisors of the number m. Then the group $\mathfrak{G}$ can be represented in the form of a product

$$\mathfrak{G} = \mathfrak{P}_1 \cdot \mathfrak{P}_2 \cdots \mathfrak{P}_k, \tag{1}$$

where $\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_k$ are certain subgroups of $\mathfrak{G}$, having as the greatest Π-Sylow divisors of their orders, respectively, the numbers

$$p_1^{\alpha_1},\ p_2^{\alpha_2}, \dots,\ p_k^{\alpha_k}.$$

There the question was also posed of the possibility of choosing pairwise permutable factors of the product (1). In the present note this question is answered affirmatively (see Theorem 1), and the theorem of S. A. Chunikhin, thus refined, is extended to a certain class of infinite groups.

Definitions. A subgroup $\mathfrak{H}$ of a group $\mathfrak{G}$ will be called Π-closed, or closed relative to the set of prime numbers Π, if every Sylow p-subgroup, $p \in \Pi$, of the subgroup $\mathfrak{H}$ is a Sylow subgroup in the group $\mathfrak{G}$. In particular, if Π coincides with the set $\Pi(\mathfrak{H})$ of prime divisors of the orders of the elements of the subgroup $\mathfrak{H}$, then the latter will be called closed in the group $\mathfrak{G}$.

A collection of pairwise permutable subgroups

$$\mathfrak{P}_1,\ \mathfrak{P}_2, \dots, \mathfrak{P}_n, \dots$$

of a locally finite group $\mathfrak{G}$ will be called separable relative to the set of prime numbers Π, or a Π-separable basis of the group $\mathfrak{G}$, if the following conditions are satisfied: 1) $\mathfrak{G} = \{\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_n, \dots\}$; 2) the subgroup $\mathfrak{P}_n$, $n = 1, 2, \dots$, is closed relative to the set $\Pi \cap \Pi(\mathfrak{P}_n)$; 3) if

$$\Pi \cap \Pi(\mathfrak{G}) = \{p_1,\ p_2, \dots, p_n, \dots\},$$

then

$$\Pi \cap \Pi(\mathfrak{P}_n) = \{p_n\}$$

and

$$\Pi \cap \Pi(\mathfrak{P}_i) \neq \Pi \cap \Pi(\mathfrak{P}_j) \quad \text{for } i \neq j.$$

Theorem 1. A finite Π-separable group possesses at least one Π-separable basis.

Proof. Suppose that the theorem is not true for all finite Π-separable groups. Among all finite Π-separable groups for which the theorem does not hold, choose some $\mathfrak{G}$ of least order g. Denote by

$$\mathfrak{N} = \mathfrak{P}_1' \times \mathfrak{P}_2' \times \cdots \times \mathfrak{P}_k'$$

the maximal special invariant Π-subgroup of the group $\mathfrak{G}$, where $\mathfrak{P}_i$, $i = 1, 2, \dots, k$, is a Sylow p_i-subgroup of the group $\mathfrak{N}$,

$$p_i \in \Pi \cap \Pi(\mathfrak{G}) = \{p_1,\ p_2, \dots, p_k\},$$

and by $\mathfrak{N}_1/\mathfrak{N}$ some minimal normal divisor of the factor group $\mathfrak{G}/\mathfrak{N}$.

The following two cases are possible.

1. The order of the factor $\mathfrak{N}_1/\mathfrak{N}$ is not divisible by any prime number from the set Π. Then, by a well-known theorem of Schur, in $\mathfrak{N}_1$ there exists a subgroup $\mathfrak{A}$ of order equal to the order of the factor $\mathfrak{N}_1/\mathfrak{N}$, and moreover all sub-

the subgroups of the normal divisor $\mathfrak{N}_1$ of such order are conjugate to one another. Consequently, the index of the normalizer $\mathfrak{N}(\mathfrak{A})_{\mathfrak{N}_1}$ in the normal divisor $\mathfrak{N}_1$ is equal to the index of the normalizer $\mathfrak{N}(\mathfrak{A})_{\mathfrak{G}}$ in the group $\mathfrak{G}$. If the index $s = [\mathfrak{G} : \mathfrak{N}(\mathfrak{A})_{\mathfrak{G}}]$ of the normalizer $\mathfrak{N}(\mathfrak{A})_{\mathfrak{G}}$ in the group $\mathfrak{G}$ is not equal to 1, then the order of the normalizer $\mathfrak{N}(\mathfrak{A})_{\mathfrak{G}}$ is less than g, and hence, by the induction hypothesis, the group $\mathfrak{N}(\mathfrak{A})_{\mathfrak{G}}$ has at least one Π-separable basis $\mathfrak{P}_1'', \mathfrak{P}_2'', \dots, \mathfrak{P}_k''$.

Putting $\mathfrak{P}_i = \mathfrak{P}_i' \cdot \mathfrak{P}_i''$, $i = 1, 2, \dots, k$, we obtain a Π-separable basis of the group $\mathfrak{G}$: $\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_k$.

Indeed, the permutability of the subgroups $\mathfrak{P}_i$ and $\mathfrak{P}_j$ follows from the permutability, between themselves, of the subgroups $\mathfrak{P}_i', \mathfrak{P}_i'', \mathfrak{P}_j', \mathfrak{P}_j''$. Since the order of the product $\mathfrak{N}(\mathfrak{A})_{\mathfrak{G}} \cdot \mathfrak{N}_1$ is equal to the product of the orders of the subgroups $\mathfrak{N}(\mathfrak{A})_{\mathfrak{G}}$ and $\mathfrak{N}_1$, equal respectively to g/s and n, divided by the order n/s of the intersection $\mathfrak{N}(\mathfrak{A})_{\mathfrak{G}} \cap \mathfrak{N}_1 = \mathfrak{N}(\mathfrak{A})_{\mathfrak{N}_1}$, i.e. is equal to g, we have the equality $\mathfrak{N}(\mathfrak{A})_{\mathfrak{G}} \cdot \mathfrak{N}_1 = \mathfrak{G}$. Hence we obtain $\mathfrak{G} = \{\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_k\}$.

Since $\mathfrak{P}_i'$ is an invariant p_i-subgroup in the group $\mathfrak{G}$, and the order of $\mathfrak{P}_i''$ can be divisible by only one prime number p_i from the set Π, the order of the subgroup $\mathfrak{P}_i = \mathfrak{P}_i' \cdot \mathfrak{P}_i''$ can be divisible by only one prime number p_i from the set Π, and therefore $\Pi \cap \Pi(\mathfrak{P}_i) = \{p_i\}$. Hence also follows the closedness of the subgroups $\mathfrak{P}_i$ with respect to $\Pi \cap \Pi(\mathfrak{P}_i)$. Thus the group $\mathfrak{G}$ has a Π-separable basis $\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_k$, which contradicts the assumption concerning this group.

If now $[\mathfrak{G} : \mathfrak{N}(\mathfrak{A})_{\mathfrak{G}}] = 1$, then the subgroup $\mathfrak{A}$ is a normal divisor of the group $\mathfrak{G}$. The order of the factor group $\mathfrak{G}/\mathfrak{A}$ is less than g, and therefore, in view of the induction hypothesis, the factor group $\mathfrak{G}/\mathfrak{A}$ has a Π-separable basis $\mathfrak{P}_1/\mathfrak{A}, \mathfrak{P}_2/\mathfrak{A}, \dots, \mathfrak{P}_k/\mathfrak{A}$.

Since, obviously, $\mathfrak{P}_i \cdot \mathfrak{P}_j = \mathfrak{P}_j \cdot \mathfrak{P}_i$, $\mathfrak{G} = \mathfrak{P}_1 \cdot \mathfrak{P}_2 \cdots \mathfrak{P}_k$, and the subgroup $\mathfrak{P}_i$, $i = 1, 2, \dots, k$, is closed with respect to $\Pi \cap \Pi(\mathfrak{P}_i)$, while $\Pi \cap \Pi(\mathfrak{P}_i) = \{p_i\}$, the collection of subgroups $\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_k$ forms a Π-separable basis of the group $\mathfrak{G}$, which contradicts the assumption about this group.

2. The order of the factor $\mathfrak{N}_1/\mathfrak{A}$ is divisible by some prime number of the set Π, for example by p_1. Since the factor $\mathfrak{N}_1/\mathfrak{A}$ has no proper characteristic subgroups and therefore decomposes into a direct product of mutually isomorphic simple groups, it follows from the Π-separability of the group $\mathfrak{G}$ that the orders of the groups $\mathfrak{P}'_2 \times \mathfrak{P}'_3 \times \dots \times \mathfrak{P}'_k$ and $\mathfrak{N}_1/\mathfrak{P}'_2 \times \mathfrak{P}'_3 \times \dots \times \mathfrak{P}'_k$ are relatively prime.

By Schur' s theorem it follows from this that in $\mathfrak{N}_1$ there exists a subgroup $\mathfrak{A}$ of order equal to the order of the group $\mathfrak{N}_1/\mathfrak{P}'_2 \times \mathfrak{P}'_3 \times \dots \times \mathfrak{P}'_k$, and all subgroups of the group $\mathfrak{N}_1$ of such order are conjugate to one another. If $[\mathfrak{G} : \mathfrak{N}(\mathfrak{A})_{\mathfrak{G}}] \neq 1$, then the order of $\mathfrak{N}(\mathfrak{A})_{\mathfrak{G}}$ is less than g, and consequently, by the induction hypothesis, the normalizer $\mathfrak{N}(\mathfrak{A})_{\mathfrak{G}}$ has a Π-separable basis $\mathfrak{P}''_1, \mathfrak{P}''_2, \dots, \mathfrak{P}''_k$.

As in case 1, from the assumption $[\mathfrak{G} : \mathfrak{N}(\mathfrak{A})_{\mathfrak{G}}] \neq 1$ it follows that the collection of subgroups $\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_k$, where $\mathfrak{P}_i = \mathfrak{P}'_i \cdot \mathfrak{P}''_i$, $i = 1, 2, \dots, k$, forms a Π-separable basis of the group $\mathfrak{G}$, which contradicts the assumption concerning this group.

Suppose now that $\mathfrak{A}$ is a normal divisor of the group $\mathfrak{G}$. In this case $\mathfrak{A}$ cannot be a p_1-subgroup, since this would contradict the maximality of the special normal divisor $\mathfrak{N}$. Moreover, for the same reason, the Sylow p_1-subgroup $\mathfrak{B}$ of the normal divisor $\mathfrak{A}$ cannot be invariant. Consequently, the order of the normalizer $\mathfrak{N}(\mathfrak{B})_{\mathfrak{G}}$ is less than g, and therefore, in view of the induction hypothesis, the subgroup $\mathfrak{N}(\mathfrak{B})_{\mathfrak{G}}$ has a Π-separable basis $\mathfrak{P}''_1, \mathfrak{P}_2, \dots, \mathfrak{P}_k$.

Putting $\mathfrak{P}_1 = \mathfrak{P}''_1 \cdot \mathfrak{A}$, it is easy to show that the collection of subgroups $\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_k$ forms a Π-separable basis of the group $\mathfrak{G}$.

Indeed, the permutability of the subgroups $\mathfrak{P}_i$ and $\mathfrak{P}_j$ is obvious, and the relation $\mathfrak{G} = \mathfrak{P}_1 \cdot \mathfrak{P}_2 \cdots \mathfrak{P}_k$ follows from the equality $\mathfrak{G} = \mathfrak{A} \cdot \mathfrak{N}(\mathfrak{B})_{\mathfrak{G}}$. By virtue of the equality of indices

$$[\mathfrak{G} : \mathfrak{N}(\mathfrak{B})_{\mathfrak{G}}] = [\mathfrak{A} : \mathfrak{N}(\mathfrak{B})_{\mathfrak{A}}]$$

the order of $\mathfrak{N}(\mathfrak{B})_{\mathfrak{G}}$ is divisible by the number

$$p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

where α_i is the highest power to which the prime number p_i occurs in g. Hence it follows that the subgroups $\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_k$ are closed in the group $\mathfrak{G}$ with respect to the corresponding sets $\Pi \cap \Pi(\mathfrak{P}_1''), \Pi \cap \Pi(\mathfrak{P}_2), \dots, \Pi \cap \Pi(\mathfrak{P}_k)$, and consequently (since $\Pi \cap \Pi(\mathfrak{A}) = \{p_1\}$), that the subgroups $\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_k$ are closed with respect to the sets $\Pi \cap \Pi(\mathfrak{P}_i)$, $i = 1, 2, \dots, k$. A contradiction to the assumption concerning the group $\mathfrak{G}$ has again been obtained. Theorem 1 is proved.

Theorem 2. *A locally* Π*-separable group* $\mathfrak{G}$*, i.e. a locally finite group every finite subgroup of which is* Π*-separable, possessing a normal series with finite factors and satisfying the minimal condition for subgroups, has at least one* Π*-separable basis.*

Proof. Since the group $\mathfrak{G}$ is a finite extension of a maximal complete normal divisor $\mathfrak{A}$ (see 2, Theorem 2), it can be represented in the form of a product $\mathfrak{G} = \mathfrak{A} \cdot \mathfrak{B}$, where $\mathfrak{B}$ is some finite subgroup. By virtue of Theorem 1 the finite group $\mathfrak{B}$ has a Π-separable basis $\mathfrak{P}_1', \mathfrak{P}_2', \dots, \mathfrak{P}_k'$, and if the prime number $p_i \in \Pi \cap \Pi(\mathfrak{G}) = \{p_1, p_2, \dots, p_k\}$ does not divide the order of the group $\mathfrak{B}$, then the subgroup $\mathfrak{P}_i'$ coincides with the identity.

Next denote by $\mathfrak{P}_1'', \mathfrak{P}_2'', \dots, \mathfrak{P}_{k-1}''$, respectively, the Sylow p_i-subgroups, $i = 1, 2, \dots, k-1$, of the group $\mathfrak{A}$, and by $\mathfrak{P}_k''$ the Sylow Π'-subgroup of this group, where Π' is the set of prime divisors of the orders of elements of the group $\mathfrak{A}$ distinct from the numbers $p_1, p_2, \dots, p_{k-1}$. From the commutativity of the group $\mathfrak{A}$ it follows that the collection of subgroups $\mathfrak{P}_1'', \mathfrak{P}_2'', \dots, \mathfrak{P}_k''$ forms its Π-separable basis. But then, putting $\mathfrak{P}_i = \mathfrak{P}_i' \cdot \mathfrak{P}_i''$, we obtain, as is easy to show, a Π-separable basis of the group $\mathfrak{G}$: $\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_k$.

Theorem 2 is proved.

Let us note that Theorem 3 of the work (^{3}) is a special case of Theorem 2.

In conclusion I take this opportunity to express my gratitude to Prof. N. S. Chernikov for his suggestions.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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