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Abstract

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MATHEMATICS

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THE CONVEXITY PROPERTY OF RIEMANNIAN SPACES OF POSITIVE CURVATURE

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For Riemannian spaces R^n of positive (negative) curvature the following property is known: the angles of every sufficiently small triangle made of shortest arcs are not less (not greater) than the corresponding angles of a plane triangle with sides of the same length (see, for example, ([9]), p. 209). A. D. Aleksandrov proved that the same assertion is true for arbitrary triangles in two-dimensional manifolds, without assumptions of smoothness ([1]). In the n-dimensional case the corresponding result "in the large" for R^n of negative curvature was proved by É. Cartan ([3]). H. Busemann generalized Cartan' s results to the so-called G-spaces of negative curvature ([2]). In the present article the following theorem is proved:

Theorem. *Let R^n be a twice continuously differentiable Riemannian manifold with a complete metric of nonpositive curvature. Then every triangle composed of shortest arcs in R_n has angles not less than the corresponding angles of a plane triangle with sides of the same length.*

The theorem is proved in the equivalent form of A. D. Aleksandrov' s "convexity condition" ([1]), p. 109).

Let us first assume that R^n has an analytic metric of strictly positive curvature. We shall denote by $(ABC)'$ the plane triangle with sides equal to the sides of the triangle ABC, formed by shortest arcs in R^n.

Lemma 1. *Let shortest arcs AB_0, AB_n $(n = 1, 2, ...)$, and B_0C be given, $B_n \in B_0C$, $B_n \neq B_0$. If AB_n converge to AB_0, and B_0 is not a point conjugate to A on AB_0, then there exists an N such that for $n \geq N$ the angles AB_0B_n and $AB_{nB}0$ are not less than the corresponding angles of the triangle $(AB_0B_n)'$.*

Proof. Introduce on AB_0 a parameter t $(0 \leq t \leq 1)$, proportional to length: $Y = Y(t)$, $A = Y(1)$. Let $\xi^i(0)$ be the unit tangent vector to B_0C at the point B_0; $\xi^i(t)$ the vector obtained from $\xi^i(0)$ by parallel displacement along B_0A. Passing through each point $Y(t)$ a geodesic in the direction $\xi^i(t)$, we obtain a surface Γ. It is not difficult to calculate that the curvature of Γ at the points of AB_0 is equal to the curvature of R^n in the two-dimensional direction tangent to Γ; therefore a sufficiently narrow strip $\Gamma_0 \subset \Gamma$ around AB_0 has positive curvature. It can be shown that, starting with some n, shortest arcs $\overline{AB_n}$, converging to AB_0, exist on Γ_0. According to A. D. Aleksandrov' s theorem for the two-dimensional case, the angle B_0 of the triangle $\overline{AB_0B_n}$ is not less than the angle B_0' of the triangle $(B_0\overline{AB_n})'$ and, consequently, is not less than the angle B_0' of the triangle $(AB_0B_n)'$, since $\overline{AB_n} \geq AB_n$. The assertion concerning the angles B_n is proved analogously.

Lemma 2. Let the shortest arcs AB, CX_n, CB be given, with $C \in AB$, $X_n \in AB$, $X_n \neq B$. Denote the angle AX_{nC} by ξ_n, and the angle between AB and BC by α. If $X_n \to B$, then there exists $\lim\limits_{n\to\infty} \xi_n = \bar{\alpha}$, and $\bar{\alpha} \leq \alpha$.

Proof. Obviously, it is enough to prove that if the shortest arcs CX_n converge to the shortest arc CB, then $\bar{\alpha} \leq \alpha$. Suppose that $\bar{\alpha} > \alpha$. Then, starting with some n, on CX_n there will be a point $\bar{D}_n$ such that $\bar{D}_n \to B$, and the angle between the shortest arcs $\bar{D}_{nB}$ and BA is equal to α. We may restrict ourselves to the case when $\alpha \neq 0$, $\alpha \neq \pi$. Take on BC a point D_n such that $D_{nB} = \bar{D}_{nB}$; it is easy to see that $\bar{D}_{nX}n = D_{nX}n + o(a_n)$, where $a_n = \bar{D}_{nB}$. Comparing the lengths $\bar{C}D_{nB}$ and CB, respectively CX_n and $CD_{nX}n$, we obtain $CD_n < \bar{C}D_n \leq CD_n + o(a_n)$, whence $\bar{C}D_n = CD_n + o(a_n)$. Take on $\bar{C}D_n$ a point E_n such that $\bar{D}_{nE}n = \bar{D}_{nB}$. It is not hard to prove that

$$E_{nB} = 2\bar{D}_{nB} \sin\frac{\psi_n}{2} + o(a_n),$$

where ψ_n is the angle between CD_n and $\bar{D}_{nB}$. Hence

$$CE_{nB} = \bar{C}D_n + \bar{D}_{nB} - 2\bar{D}_{nB} + E_{nB} = CD_n + D_{nB} + 2\bar{D}_{nB}\left(1 - \sin\frac{\psi_n}{2}\right) +$$

$$+o(a_n) = CB - 2D_{nB}\left(1 - \sin\frac{\psi_n}{2}\right) + o(a_n).$$

Since $\lim\limits_{n\to\infty} \psi_n = \pi - (\bar{\alpha} - \alpha) < \pi$, for sufficiently large n we have $CE_{nB} < CB$; but this is impossible, since CB is a shortest arc.

The proof of the theorem can now be carried out according to the plan of the proofs of A. D. Aleksandrov (([1]), pp. 100-103). Let OP and OQ be shortest arcs; Y a fixed interior point of OP; X a variable point of OQ; $OX = x$, $OY = y$; $\gamma(x, y)$ the angle at O' of the triangle $(OXY)'$; $\alpha(x)$ the angle between any shortest arc YX and XO. It is enough to prove that $\gamma(x, y)$ is a nonincreasing function of x. Using the analyticity of the metric of R^n, one can show that by an arbitrarily small shift OQ may be replaced by a shortest arc $\overline{OQ}$ having the following property: on $\overline{OQ}$ there exists only a finite number M of points X conjugate to Y on some one of the shortest arcs YX. We shall assume that OQ already has this property.

Two assertions are proved simultaneously:

1. $\gamma(x, y)$ is a nonincreasing function of x.
2. $\alpha(x)$ is not smaller than the corresponding angle of the triangle $(OXY)'$, whatever the shortest arc XY.

By Lemma 1, assertions 1 and 2 are true for sufficiently small x. Let x_0 be the least upper bound of those x for which 1 and 2 are true; it is required to prove that $x_0 = OQ$. Suppose, on the contrary, that $x_0 < OQ$. As $X \to X_0$, $x < x_0$, the assumptions of Lemma 2 are fulfilled; consequently, 1 and 2 are true also for X_0. If one assumes that $X_0 \notin M$, then as $X \to X_0$, $x > x_0$, the conditions of Lemma 1 are fulfilled; considering the triangles OYX_0 and YX_0X_1 ($x_1 > x_0$) and applying the "lemma on convex quadrilaterals" (([1]), p. 100), we obtain that 1 and 2 are true also for X_1, which is impossible. Hence $X_0 \in M$. Now take any $x_1 > x_0$ such that $(X_0X_1)_1 \cap M = 0$. Applying the preceding considerations to the shortest arcs X_1Y, X_1X_0 (instead of OP, OQ), we become convinced that 1 and 2 are valid for all $X \in X_1X_0$, including X_0. Using again the lemma on convex quadrilaterals, it is easy to see that 1 and 2 are true also for X_1, which contradicts the definition of x_0. Thus, $x_0 = OQ$, as was required to prove.

Approximating a twice continuously differentiable metric of strictly positive curvature by an analytic metric, it is easy to remove the requirement that the metric be analytic.

Finally, replacing the comparison plane triangles by triangles of the Lobachevsky plane (cf. ([1]), p. 345), one can extend the theorem to the case of nonnegative curvature. The theorem is also generalized to spaces whose curvature is bounded below. Let us also note that the limit-

transition makes it possible to carry the theorem over to arbitrary complete convex surfaces in n-dimensional Euclidean space. Hence it follows:

Theorem. *On an $(n-1)$-dimensional complete convex surface (without smoothness assumptions) in n-dimensional Euclidean space, a shortest curve is uniquely determined by any sufficiently small arc of it.*

This theorem is an n-dimensional generalization of A. D. Aleksandrov's "theorem on the non-overlapping of shortest curves" ([1], p. 109).

In conclusion I express my deep gratitude to A. I. Fet for posing the problem and for his constant help in the work.

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Note: Figure translations are in progress. See original paper for figures.