Mathematics

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Abstract

Full Text

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ON THE INSUFFICIENCY OF THE METHOD OF CHARACTERISTIC EXPONENTS IN APPLICATION TO NONLINEAR EQUATIONS

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Consider a system of n differential equations in vector form

$$\frac{dx}{dt} = F(t, x), \tag{1}$$

for which $F(t, 0) \equiv 0$ and the Lipschitz condition is satisfied

$$|F(t, x_1) - F(t, x_2)| \leq K\,|x_1 - x_2|\,. \tag{2}$$

From the estimate of the norms of solutions known under these conditions,

$$|x(t)| \leq |x(0)|e^{kt},$$

it follows that the characteristic exponents of all solutions are bounded:

$$\lambda = \overline{\lim_{t\to\infty}}\, t^{-1} \ln |x(t)| \leq K,$$

and therefore there exists $\sup \lambda = \Lambda \leq K$. It is also known ([1]) that in the case of linearity of system (1), i.e., when $F(t, x) = A(t)x$, the following assertion is true:

For every $\varepsilon > 0$ there exists a constant B_ε, the same for all solutions, such that

$$|x(t)| \leq |x(0)|B_\varepsilon e^{(\Lambda+\varepsilon)t}. \tag{3}$$

In the nonlinear case, from the definitions of λ and Λ there likewise follows an inequality of the form (3), in which, however, B_ε depends on the choice of the solution $x(t)$, or, what is the same, on its initial point $x(0)$.

The question arises: is it possible also in the nonlinear case to choose B_ε independently of $x(0)$?

A priori, inequality (3) may fail for two reasons:

a) B_ε exists for each sphere $|x(0)| = r < R$, but grows without bound as $r \to 0$ (or $r \to \infty$);

b) for no sphere does B_ε exist (since from the existence of B_ε for $r = r_0$ there follows its existence for $r < r_0$).

Case a) still leaves some possibility for using the number Λ; thus, the following theorem holds:

Theorem 1. *In case a), from the negativity of Λ there follows asymptotic stability of the trivial solution $x = 0$ of system (1).*

In case b), however, the method of characteristic exponents proves unsuitable for studying solutions of system (1) in the sense in which this is done for linear systems; for example, from the condition $\Lambda < 0$ even ordinary stability of the solution $x = 0$ does not follow.

The following examples show that both cases are indeed encountered, and moreover with the right-hand side of (1) independent of t.

Example 1. The system consists of one equation with one unknown x and is given in the strip $t \geqslant 0,\ |x| \leqslant 1$:

$$\frac{dx}{dt} = \frac{2\ln^2 x}{1+\ln^2 x}\,x \quad \text{for } x \neq 0;$$

$$\frac{dx}{dt} = 0 \quad \text{for } x = 0.$$

The Lipschitz condition is satisfied, since the derivative of the right-hand side with respect to x is bounded. The general solution has the form

$$x(t) = \pm e^{t-c-\sqrt{1+(t-c)^2}},$$

so that its characteristic exponent is $\lambda = 0$. The sphere $|x(0)| = r$ consists of two points: $x(0) = \pm r$, and for them (3) is satisfied; however, the constant B_ε cannot be uniform as $r \to 0$.

Indeed, assuming (it is enough to consider $x(t) > 0$) that

$$\frac{x(t)}{x(0)} < B_\varepsilon e^{\varepsilon t},$$

we obtain on the left-hand side the expression

$$e^{t-\sqrt{1+(t-c)^2}+\sqrt{1+c^2}},$$

Fig. 1

Figure 1: Fig. 1

which for $t = c$ becomes

$$e^{c-1+\sqrt{1+c^2}},$$

whereas the right-hand side is $B_\varepsilon e^{\varepsilon c}$, and the inequality is violated for large c (i.e., for small $x(0)$), if $\varepsilon < 2$.

Fig. 1

Example 2. The system consists of two equations and is given in the whole plane (x, y):

$$\frac{dx}{dt} = \frac{x^2(y-x)+y^5}{(x^2+y^2)[1+(x^2+y^2)^2]}, \qquad \frac{dy}{dt} = \frac{y^2(y-2x)}{(x^2+y^2)[1+(x^2+y^2)^2]}.$$

and, by continuity,

$$x' = y' = 0 \quad \text{for } x = y = 0.$$

The partial derivatives of the right-hand sides do not exist at $x = y = 0$, but they are bounded on the whole plane, which ensures that the Lipschitz condition holds with a common constant for any pair of points, including $(0, 0)$. In view of the symmetry of the field of directions with respect to $(0, 0)$, it is sufficient to study the upper half-plane, including the right half-axis $x \geqslant 0$ and excluding the left one.

Combining Frommer' s method with additional investigations, one can establish the following picture of the behavior of the solutions (see Fig. 1). The system has a unique singular point $(0, 0)$; every solution enters the shaded sector in finite time and, while remaining in it, tends to $(0, 0)$, its characteristic exponent satisfying $\lambda \leqslant -a^2 < 0$. There exists an elliptic region (hatched), and every solution starting near the half-axis $x < 0$ goes around this region before entering the sector. Thus, despite the condition $\Lambda \leqslant -a^2 < 0$, the singular point is unstable, since every solution beginning in the left part of an arbitrarily small neighborhood of this point moves away from it by no less than the diameter of the elliptic region before again entering the small neighborhood. Hence, by Theorem 1, it follows that case b) occurs.

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CITED LITERATURE

1. A. M. Lyapunov, *The General Problem of the Stability of Motion*, 1950.

Note: Figure translations are in progress. See original paper for figures.