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Abstract

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MATHEMATICS

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ON CHARACTERISTIC PROPERTIES OF SOLUTIONS OF REGULAR AND QUASIREGULAR PROBLEMS OF THE CALCULUS OF VARIATIONS

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This note considers the question of characteristic properties of curves that can give an absolute minimum to regular and quasiregular positive-definite single integrals of the calculus of variations in parametric form, as well as the related question of convergence of curves with respect to the functional. Differentiability of the integrands is not assumed.

1. We shall call a function $F(P, A)$, $P \in D$, $A \in R^2$ (where D is a closed bounded set of points of the plane, and R^2 is the two-dimensional vector space), **quasiregular** if it is continuous in the totality of its arguments and satisfies the following requirements:

1) $F > 0$ for all $P \in D$ and any $A \neq 0$;

2) $F(P, kA) = kF(P, A)$ for any $k > 0$;

3) $F(P, A_1 + A_2) \leq F(P, A_1) + F(P, A_2)$.

The functional

$$I(\Gamma) = \int_a^b F(P, \dot{P})\, dt$$

(also called quasiregular) is defined and finite for every rectifiable curve Γ. Without always specifying this, we shall assume that all integrands occurring below are quasiregular and that all curves lie in the set D, which is assumed fixed.

If in condition 3) equality can be attained only when

$$|A_1 + A_2| = |A_1| + |A_2|,$$

then the function F and the functional $\int F\,dt$ are called **regular**.

2. The equality $F(P, A) = 1$, for each P, determines in R^2 a closed convex curve (the indicatrix). The indicatrix may contain rectilinear segments. We shall call the **angle of linearity** of the function F at a point P the closed part of the plane R^2 enclosed between two rays drawn from the origin to the endpoints of some rectilinear segment of the indicatrix $F(P, A) = 1$. Regular functions have no angles of linearity at any point P. Suppose that, for every point $P \in D$, the angles of linearity of the function F_1 are contained in the angles of linearity of the function F_2; then we shall use the notation $F_1 < F_2$. We shall say that the quasiregular functionals $\int F_1\,dt$ and $\int F_2\,dt$ **belong to the same class** if $F_1 < F_2$ and $F_2 < F_1$. All regular functionals belong to the same class as the functional that is the length of the curve.

3. One says that a sequence of curves Γ_n **converges to a curve** Γ_0 **with respect to the functional** $I(\Gamma)$ if $\rho(\Gamma_n, \Gamma_0) \to 0$ and $I(\Gamma_n) \to I(\Gamma_0)$ as $n \to \infty$. Here $\rho(\Gamma_1, \Gamma_2)$ denotes the distance in the sense of Fréchet.

Theorem 1. *If $F_1 < F_2$, then from the convergence of a sequence of curves with respect to the functional $\int F_1\,dt$ there follows the convergence of this sequence also with respect to $\int F_2\,dt$.*

This theorem is a generalization of Tonelli's theorem [1] stating that convergence of curves with respect to a regular functional implies convergence in length. For an extension of Tonelli' s theorem to the case of double integrals, see [2].

4. We shall call a curve Γ **smooth with respect to the functional** $I(\Gamma) = \int F\,dt$ if the ratio

$$I(\gamma) : F(P_1, P_2 - P_1)$$

(where γ is an arc of the curve Γ, and P_1 and P_2 are the initial and terminal points of the arc γ) tends to 1 when the vector $P_2 - P_1$ tends to 0. In particular, a curve will be smooth with respect to length if the ratio of the length of an arc to the length of the chord subtending this arc tends to 1 when the arc contracts to any curve. A continuously differentiable curve is smooth with respect to any quasiregular functional. The converse is false: for example, the curve $\rho = e^{-\varphi^2}$, $0 \leqslant \varphi \leqslant \infty$, is smooth with respect to length, but has no tangent at $\rho = 0$.

Let the curve Γ, without self-intersections, be smooth with respect to $\int F\,dt$. Then for every point P_0 of the curve Γ the following condition is fulfilled. Let γ_n be a sequence of arcs of the curve Γ contracting to P_0. Denote by $\bar{\gamma}_n$ the curve obtained from γ_n by a similarity transformation and a parallel translation so that

$$|P_{2n} - P_{1n}| = 1$$

and $P_{1n} = P_0$ for all n, where P_{1n} and P_{2n} are, respectively, the initial and terminal points of the arc γ_n. Then the sequence $\bar{\gamma}_n$ is compact, and every convergent subsequence of arcs $\bar{\gamma}_{n_k}$ converges to its limiting arc $\bar{\gamma}_0$ with respect to the "elementary" functional

$$\int F_0(\dot{P})\, dt,$$

where

$$F_0(A) \equiv F(P_0, A),$$

and, moreover, the limiting arc $\bar{\gamma}_0$ gives an absolute minimum to the integral $\int F_0\, dt$ in the problem with fixed endpoints.

Conversely, if for every point P_0 of some curve Γ without self-intersections this condition is satisfied, then Γ is smooth with respect to the functional $\int F\, dt$.

From this necessary and sufficient criterion for smoothness with respect to a functional, and from Theorem 1, the following theorem follows immediately.

Theorem 2. *If $F_1 < F_2$, and if a curve Γ without self-intersections is smooth with respect to $\int F_1\, dt$, then it is smooth with respect to $\int F_2\, dt$.*

Thus a curve without self-intersections can be smooth (or not smooth) only simultaneously with respect to all functionals of the given class. In particular, smoothness with respect to a regular functional is equivalent to smoothness with respect to length.

5. The connection between the condition of smoothness with respect to a functional and variational problems is established by the following theorems.

Theorem 3. *If a curve Γ gives an absolute minimum of the functional $\int F\, dt$ in the problem with fixed endpoints, then Γ is smooth with respect to $\int F\, dt$.*

Theorem 4. *Let the curve Γ_0, without self-intersections, be smooth with respect to the quasiregular functional $\int F\, dt$. Then there exists a continuous function of the point $G(P)$, with $G > 0$ everywhere in D, such that Γ_0 gives a strict absolute minimum to the functional*

$$I_1(\Gamma) = \int G(P) F(P, \dot{P})\, dt.$$

Let us note that under these assumptions the function $G(P)F(P, A)$ is quasiregular, and the functionals $\int F\, dt$ and $\int GF\, dt$ belong to one and the same class, since F and GF have identical lineality angles.

If one puts $F(P, A) \equiv A$, then from Theorem 4 it follows that for any curve smooth with respect to length (for example, $\rho = e^{-\varphi^2}$, $0 \leqslant \varphi \leqslant \infty$) one can construct an integral of geometrical optics $\int G(P)\, ds$ that assumes a minimum on this curve. It should be noted that the integrand of a regular functional attaining a minimum on a curve that is smooth with respect to length but not smooth in the ordinary sense cannot be ···

differentiable (and even satisfy the Lipschitz condition), which follows from the result of the work of Busemann and Mayer ([3]), showing that for such integrands the curve giving the minimum is smooth. For the case of semi-definite problems, a construction analogous to Theorem 4 was made by M. A. Lavrent' ev ([4]).

6. From Theorems 2, 3, 4 it follows that the characteristic property, necessary and sufficient for a given curve without self-intersections to give an absolute minimum to one of the functionals of the given class, is the smoothness of this curve with respect to some functional of this class. In particular, a curve gives an absolute minimum to one of the regular functionals if and only if it is smooth with respect to length.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.