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Abstract

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PROJECTIVELY INVARIANT FRAMES OF A RULED SURFACE BELONGING TO A GIVEN CONGRUENCE

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In the present paper we consider projectively invariant frames of a ruled surface belonging to a given congruence (analogous constructions in metric and affine geometries were carried out in (^{1}, ^{2})). The notation, for the most part, corresponds to that adopted in (^{3}, ^{4}).

§ 1. **Construction of frames.** Let A_1A_2 be a ray of the congruence, and let A_1, A_2, A_3, A_4 be the vertices of a frame. In the derivation formulas $dA_i = \omega_i^k A_k$, the Pfaff forms $\omega_i^k = a_i^k\omega_1^3 + b_i^k\omega_1^4$ satisfy the well-known structure equations $D\omega_i^k = [\omega_i^j\omega_j^k]$. Taking ω_1^3 and ω_1^4 as the fundamental forms and carrying out the first step of fixing the frame, we obtain: $\omega_2^3 = \omega_1^4$, $\omega_2^4 = \omega_1^3$. In this case the foci F_i of the ray are determined by the equalities $F_1 = A_1 + A_2$, $F_2 = A_1 - A_2$, and the torses of the congruence by the equation $(\omega_1^3)^2 - (\omega_1^4)^2 = 0$.

Two ruled surfaces S_1 and S_2, passing through the given ray of the congruence and conjugate in the sense of Sannia (^{5}), can be characterized in projective-differential geometry by the following property: the tangent plane of the surface S_1 at any point M_1 of the ray is the tangent plane of the surface S_2 at the point M_2 of the ray harmonically conjugate to M_1 with respect to the foci of the ray; moreover, the tangent plane of the surface S_1 at M_2 is the tangent plane of the surface S_2 at M_1.

With our fixation the equation $\omega_1^3\omega_1^4 = 0$ singles out the conjugate net of ruled surfaces belonging to the congruence, while the planes $A_1A_2A_3$ and $A_1A_2A_4$ are the tangent planes of these surfaces at the points A_1 and A_2.

At the second step of fixing, two variants arise. In both variants the points $G_1 = A_3 + A_4$ and $G_2 = A_3 - A_4$ fall on the rays of the congruences K_1 and K_2, which are the Laplace transforms of the given congruence by means, respectively, of the first and second focal nets; and a normalization is carried out that excludes from consideration ruled surfaces for which at least one focal line degenerates.

In the first variant (the K-frame) the fixation is performed so that the lines A_1A_4 and A_2A_3 belong to the osculating complex of the net for the given ray (i.e. to the complex of lines determined by the given ray and by two pairs of infinitely close rays of ruled surfaces of the net passing through this ray), while in the second (the M-frame) it is performed so that the vertices are the median (^{6}) points of the ray with respect to the foci.

The last step of the fixation is carried out so that the line A_3A_4 becomes the line joining the second foci of the rays of the congruences K_1 and K_2 (these foci coincide with the points G_1 and G_2). Adding also the usual normalization $(A_1A_2A_3A_4) = 1$, we obtain the following relations for the coefficients a_i^k and b_i^k:

$$b_1^3 = a_1^4 = a_2^3 = b_2^4 = 0, \qquad a_1^3 = b_1^4 = b_2^3 = a_2^4 = 1,$$

$$a_1^1 + a_2^2 + a_3^3 + a_4^4 = b_1^1 + b_2^2 + b_3^3 + b_4^4 = 0,$$

$$\begin{aligned} a_1^2 - a_2^1 = b_1^1 - b_2^2 &= A, & b_1^2 - b_2^1 = a_1^1 - a_2^2 &= B, \\ a_4^3 - a_3^4 = b_3^3 - b_4^4 &= A^*, & b_4^3 - b_3^4 = a_3^3 - a_4^4 &= B^*, \\ a_3^2 - a_4^1 &= b_1^2 - b_3^1, & b_3^2 - b_4^1 &= a_4^2 - a_3^1, \\ B^2 - A^2 = c &= \text{const}, & A + \varepsilon A^* &= 0, \end{aligned}$$

where $\varepsilon = +1$ for the K-frame and $\varepsilon = -1$ for the M-frame.

Taking these relations into account, the integrability conditions of the derivational formulas will contain 11 exterior differential equations in 14 functions and will determine a congruence referred to an arbitrary net of conjugate ruled surfaces, with an arbitrariness of 3 functions of 2 arguments.

Putting $\omega_1^4 = 0$, $(\omega_1^3)_{\omega_1^4=0} = ds$, $(a_i^k)_{\omega_1^4=0} = \alpha_i^k$, we obtain 2 frames of a ruled surface belonging to the given congruence. The derivational formulas of these frames will have the form:

$$dA_i/ds = \alpha_i^k A_k, \tag{1}$$

$$\alpha_1^4 = \alpha_2^3 = 0, \qquad \alpha_1^3 = \alpha_1^4 = 1, \qquad (\alpha_1^1 - \alpha_2^2)^2 - (\alpha_1^2 - \alpha_2^1)^2 = c = \text{const},$$

$$\alpha_1^2 - \alpha_2^1 = \varepsilon(\alpha_3^4 - \alpha_4^3), \qquad \alpha_1^1 + \alpha_2^2 + \alpha_3^3 + \alpha_4^4 = 0,$$

$$(\alpha_3^3 - \alpha_4^4)^2 - (\alpha_4^3 - \alpha_3^4)^2 = cI, \tag{2}$$

where I is a known invariant of the congruence (the Welsch invariant). Since the frame is completely fixed, ds and α_i^k are projective invariants of the ruled surface belonging to the given congruence.

§ 2. **Computational formulas.** Consider the frame of a congruence, referred to the torses and Laplace transformations, introduced in (^{7}), Ch. VI, § 4. Its derivational formulas may be written, in the notation of §§ 185–191 (^{4}) (replacing only A_i and ω_i^k by B_i and v_i^k), in the form: $dB_i = v_i^k B_k$, where v_1^3 and v_2^4 are the principal forms, and the remaining v_i^k are expressed through them by means of the Fubini coefficients $\alpha, \alpha', \beta, \beta', \gamma$, etc.; moreover $\beta = \beta' = \beta_2 = \beta_1' = 0$ by virtue of the choice of the vertices B_3 and B_4, and $v_1^1 + v_2^2 + v_3^3 + v_4^4 = 0$ by virtue of the normalization $(B_1 B_2 B_3 B_4) = 1$. This frame is connected with our frames by the formulas:

$$B_1 = \lambda_1(A_1 + A_2), \qquad B_2 = \lambda_2(A_1 - A_2),$$

$$B_3 = \lambda_3(A_3 + A_4), \qquad B_4 = \lambda_4(A_3 - A_4),$$

where

$$4\lambda_1\lambda_2\lambda_3\lambda_4 = 1, \qquad 4\gamma\gamma'\lambda_1\lambda_2 = c\lambda_3\lambda_4, \qquad \gamma'\lambda_1^3\lambda_4^2\lambda_3 - \gamma\lambda_2^3\lambda_3^2\lambda_4 + \varepsilon(\alpha\lambda_1^2\lambda_2\lambda_4^3 - \alpha'\lambda_1\lambda_2^2\lambda_3^3) = 0.$$

From this one obtains the following formulas for computing α_i^k:

$$c\,ds^2 = 4\gamma\gamma' v_1^3 v_2^4, \qquad 4\alpha_1^1\,ds = \psi_1 + 2\Omega q, \qquad 4\alpha_2^2\,ds = \psi_1 - 2\Omega q,$$

$$4\alpha_3^3 ds = -\psi_1 + 2\Omega_1 q, \qquad 4\alpha_4^4 ds = -\psi_1 - 2\Omega_1 q,$$

$$4\alpha_1^2 ds = d\ln\varphi_2 v_2^4 - d\ln\varphi_1 v_1^3 + 2(v_1^1 - v_2^2 - \varepsilon\Omega_2 q),$$

$$4\alpha_2^1 ds = d\ln\varphi_2 v_2^4 - d\ln\varphi_1 v_1^3 + 2(v_1^1 - v_2^2 + \varepsilon\Omega_2 q),$$

$$4\alpha_3^4 ds = d\ln\varphi_2 v_1^3 + 2(v_3^3 - v_4^4 - \Omega_2 q) - d\ln\varphi_1 v_2^4,$$

$$4\alpha_4^3 ds = d\ln\varphi_2 v_1^3 + 2(v_3^3 - v_4^4 + \Omega_2 q) - d\ln\varphi_1 v_2^4,$$

$$8\alpha_3^1 ds^2 = 4\psi_2 + c\psi_3 q\,ds^2, \qquad 8\alpha_3^2 ds^2 = 4\psi_2^* + c\psi_3^* q\,ds^2,$$

$$8\alpha_4^1 ds^2 = 4\psi_2^* - c\psi_3^* q\,ds^2, \qquad 8\alpha_4^2 ds^2 = 4\psi_2 - c\psi_3 q\,ds^2,$$

where

$$\psi_1 = \gamma_1' v_1^3 + \gamma_2 v_2^4, \quad \varphi_1 = \gamma (v_2^4)^2 - \varepsilon\alpha (v_1^3)^2, \quad \varphi_2 = \gamma' (v_1^3)^2 - \varepsilon\alpha' (v_2^4)^2,$$

$$\psi_2 = v_1^3 v_3^1 + v_1^4 v_4^3, \quad \psi_2^* = v_1^3 v_3^1 - v_2^4 v_4^3, \quad \gamma\gamma'\psi_3 = \alpha\beta_1\gamma'(v_1^3)^3 - \varepsilon\alpha\alpha'(\beta_1 v_2^4 + \beta_2' v_1^3) v_1^3 v_2^4 + \gamma\alpha'\beta_2'(v_2^4)^3,$$

$$\begin{gathered}\gamma\gamma'\psi_3^* = \alpha'\beta_2'\gamma\,(v_2^4)^3 + \alpha\alpha'(\beta_1 v_2^4 - \beta_2' v_1^3) v_1^3 v_2^4 - \alpha\beta_1\gamma'(v_1^3)^3,\\ \Omega = 2\gamma\gamma'(v_1^3)^2(v_2^4)^2 - \varepsilon[\alpha\gamma'(v_1^3)^4 + \alpha'\gamma(v_2^4)^4],\\ \Omega_1 + \Omega_2 = 2\alpha(v_1^3)^2\varphi_2, \qquad \Omega_1 - \Omega_2 = 2\alpha'(v_2^4)^2\varphi_1\end{gathered}$$

are relatively invariant differential forms and $q = (v_1^3 v_2^4 \varphi_1 \varphi_2)^{-1/2}$.

These formulas show that the invariants $\alpha_1^2, \alpha_2^1, \alpha_3^4, \alpha_4^3$ are of the second order, while the remaining α_i^k are of the first order, and also that in the K-frame there are excluded from consideration the surfaces $(\varphi_1\varphi_2)_{\varepsilon=1} = 0$, along which the line F_1G_1 (or F_2G_2) coincides with an asymptotic tangent, whereas in the M-frame there are excluded from consideration the surfaces $(\varphi_1\varphi_2)_{\varepsilon=-1} = 0$, for which one of the focal lines is asymptotic.

§ 3. **Geometric meaning of the invariants.** The geometric characterization of the elements of our frames is clear from § 1. We shall call the vertices A_1 and A_2 the projective centers of the ray. Consider the points X_1^i, X_2^i, X_3^i, X_4^i $(i = 1, 2)$ of intersection of the ray A_1A_2 with the planes of the pencil having axis A_3A_4 and passing, respectively, through:

1) points infinitely near to the foci of the ray on the given ruled surface S;
2) points infinitely near to the centers of the ray on S;
3) points infinitely near to the centers of the ray on a ruled surface passing through the ray, for which $\alpha_1^2 = 0$ or $\alpha_2^1 = 0$, and which has two common nearby rays with S;
4) tangents to the lines described by the vertices A_3 and A_4 of the frame.

We shall denote by Y^i the points of intersection of the line A_3A_4 with the planes of the pencil having axis the ray A_1A_2 and passing through points infinitely near to the points A_i $(i = 3, 4)$. Then we obtain the following formulas, showing the geometric meaning of our invariants:

$$\begin{aligned} c\,ds^2 &= 4DV_0(F_1F_2X_1^1X_2^1), & \alpha_1^2 ds &= DV_0(A_2A_1F_1X_2^1),\\ \alpha_2^1 ds &= DV_0(A_1A_2F_1X_2^2), & \alpha_3^4 ds &= DV_0(A_4A_3G_1Y^3),\\ \alpha_4^3 ds &= DV_0(A_3A_4G_1Y^4), & \alpha_3^2 ds^2 &= 2DV(A_2A_1F_1X_3^1),\\ \alpha_4^1 ds^2 &= 2DV_0(A_1A_2F_1X_3^2), & \alpha_3^1 &= \alpha_3^2 DV(A_1A_2F_1X_4^1),\\ \alpha_4^2 &= \alpha_4^1 DV(A_2A_1F_1X_4^2), \end{aligned}$$

where DV denotes the compound ratio, and DV_0 its principal part.

We shall call E-surfaces the ruled surfaces belonging to the given congruence and giving an extremum of the invariant ds. Denote by $\bar{\alpha}_i^k, \overline{ds}$ the invariants of the E-surface conjugate to S. Then we shall have

$$(\alpha_1^1 + \alpha_2^2)\, ds = \begin{cases} (\bar{\alpha}_1^2 - \bar{\alpha}_3^4)\, \overline{ds}, & \text{for } \varepsilon = 1, \\ (\bar{\alpha}_1^2 - \bar{\alpha}_4^3)\, \overline{ds}, & \text{for } \varepsilon = -1. \end{cases}$$

These formulas, together with (2), make it possible to determine the geometric meaning of the invariants α_i^i.

§ 4. **Natural equations.** The results obtained in §§ 1 and 3 make it possible to find the natural equations of projective-invariant classes of ruled surfaces belonging to the given congruence. We indicate some of them: $\alpha_1^2 = 0$—surfaces on which the tangent to the first line of centers (A_1) coincides with the edge of the frame A_1A_3; $\alpha_3^4 = 0$—surfaces for which A_1A_3 describes a torse whose cuspidal edge does not coincide with the line of centers; $\alpha_3^1 = 0$—surfaces for which the plane $A_2A_3A_4$ contains the tangent to the line (A_3); $\alpha_3^2 = 0$—surfaces for which the plane $A_1A_3A_4$ contains the tangent to the line (A_3); $\alpha_1^2 + \alpha_3^4 = 0$ —surfaces for which the first line of centers

is asymptotic (we note that in the case of an M-frame both lines of centers become asymptotic simultaneously); $\alpha_1^2 = \alpha_3^4$—surfaces for which A_1A_3 is an asymptotic tangent (in the case of a K-frame, A_1A_3 and A_2A_4 possess this property simultaneously); $\alpha_4^3 - \alpha_3^4 \pm (\alpha_2^2 + \alpha_4^4) = 0$ in a K-frame—surfaces one of whose focal lines is asymptotic; $\alpha_4^3 - \alpha_3^4 \pm (\alpha_4^4 + \alpha_1^1) = 0$ in an M-frame—surfaces for which the line F_1G_1 (or F_2G_2) is an asymptotic tangent; $\alpha_4^2\alpha_4^3 - \alpha_4^2 = 0$—surfaces for which A_1A_4 describes a torse; $\alpha_3^1\alpha_4^2 - \alpha_3^2\alpha_4^1 = 0$—surfaces for which the line G_1G_2 describes a torse.

Similarly characterized are the surfaces $\alpha_2^1 = 0,\ \alpha_4^3 = 0,\ \alpha_4^2 = 0,\ \alpha_4^1 = 0$, etc. The differential equations of all these surfaces (in torse parameters) are easily obtained from the formulas of § 2.

§ 5. **Egorov transformations.** Let $v = \text{const}$ be a one-parameter family of ruled surfaces belonging to the given congruence and referred to a K- or M-frame. Let $M \to P(M)$ be a projective transformation whose coefficients depend on the parameter v. We shall seek such families of ruled surfaces belonging to the congruence for which one can find a transformation $M \to P(M)$ under which one of the focal surfaces of the congruence A_1A_2 is transformed into the focal surface of the congruence $P(A_1)P(A_2)$ (such transformations we shall call —by analogy with (1) and (2))—**projective Egorov transformations of the first kind**).

The natural equation of the required class (containing ∞^{15} surfaces) can be written in the form*

$$\det \|q_i^k\| = 0. \tag{3}$$

Here $q_1^1 = q_1^6 = 1,\ q_1^2 = q_1^5 = \pm 1$, and the remaining $q_1^k = 0$. To find $q_2^k, \dots, q_{15}^k$ we obtain the recurrence formulas

$$q_{i+1}^k = \frac{d}{ds} q_i^k + q_i^\alpha \beta_\alpha^k;$$

β_α^k are the coefficients of the derivative formulas, obtained from (1), $dR_\alpha/ds = \beta_\alpha^k R_k$, where

$$R_1 = (P'(A_1),\ P(A_1),\ P(A_2),\ P(A_3)), \quad R_2 = (P'(A_1),\ P(A_1),\ P(A_2),\ P(A_4))$$

and so on, while P' denotes the projective transformation whose coefficients are the derivatives of the coefficients of the transformation P.

In an analogous way one finds the surfaces by means of which Egorov transformations of the second kind are effected (preserving the conjugacy of the net of ruled surfaces) and of the third kind (the projective centers remain points harmonically separating the foci). In the first case, in the first row of equation (3), the nonzero entries are $q_1^2 = -q_1^5 = 1$; in the second case $q_1^2 = q_1^5 = 1$.

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* All indices run through the values from 1 to 15.

Note: Figure translations are in progress. See original paper for figures.

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