MATHEMATICS

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Abstract

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LIE QUADRIC FOR RULED SURFACES OF A CONGRUENCE

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1. In 1878 Sophus Lie introduced the concept of an osculating surface of the second order. This surface is the limiting position of a surface of the second order passing through the tangents to three asymptotic lines of one family at the points of their intersection with one asymptotic line of the other family, when these points approach one another and in the limit coincide. Subsequently this surface was used for the construction of the normal tetrahedron of a surface ([1]).

In the present work an invariant equation (with respect to all tetrahedra of the first order) is obtained for the Lie quadric of ruled surfaces of a congruence, and certain questions connected with this surface are considered. The method of Cartan' s exterior forms ([2]) is used in the work.

2. An infinitesimal displacement of the tetrahedron $A_1A_2A_3A_4$ in projective space is determined by the system of differential equations

$$dA_i = \omega_i^k A_k \qquad (i, k, j = 1, 2, 3, 4), \tag{1}$$

where ω_i^k are linear differential forms connected by the structure equations of projective space

$$D\omega_i^k = [\omega_i^j \omega_j^k].$$

The family of tetrahedra of the first order attached to the congruence is singled out by the differential equations

$$\omega_1^4 = 0, \qquad \omega_2^3 = 0.$$

Differentiating these equations exteriorly, we obtain, with the aid of the structure equations, two quadratic equations

$$[\omega_1^2\omega_2^4] + [\omega_1^3\omega_3^4] = 0, \qquad [\omega_2^1\omega_1^3] + [\omega_2^4\omega_4^3] = 0.$$

Expanding them into linear equations by means of Cartan' s lemma, we obtain

$$\omega_3^4 = \alpha\omega_1^3 - \beta\omega_2^4, \qquad \omega_2^1 = \gamma'\omega_1^3 + \beta'\omega_2^4, \tag{2}$$

$$\omega_1^2 = \beta\omega_1^3 + \gamma\omega_2^4, \qquad \omega_4^3 = -\beta'\omega_1^3 + \alpha'\omega_2^4.$$

3. Let an arbitrary ruled surface of the congruence be defined by the equation $\omega_2^4 = \lambda\omega_1^3$. Then on its osculating Lie surface (or Lie quadric) there will lie the ray A_1A_2 and two of its infinitely close rays. Consequently, the Lie surface may be determined in a five-

in the projective space P_5 by the points [12], $d[12]$ $(\mathrm{mod}\,(\omega_2^4 = \lambda\omega_1^3))$ and $d^2[12]$ $(\mathrm{mod}(\omega_2^4 = \lambda\omega_1^3))$, where $[ik]$ is the analytic line A_iA_k.

The two-dimensional plane $([12]d[12]d^2[12])$ intersects the hyperquadric Q_2^4 in a curve whose points will be the images of one series of rectilinear generators of the required surface of Lie.

In view of equations (1) and (2), transforming the plane $([12]d[12]d^2[12])$ mod $(\omega_2^4 = \lambda\omega_1^3)$, we obtain:

$$\{[12],\ \lambda[14]-[23],\ 2\lambda[34]+\lambda(\lambda_1+\lambda\lambda_2)[14]+$$
$$+(\lambda^2\alpha' - 2\lambda\beta' - \gamma')[13] + (-\lambda^2\gamma - 2\lambda\beta + \alpha)[42]\},$$

where

$$\Delta\lambda = d\ln\lambda + \omega_1^1 + \omega_4^4 - \omega_2^2 - \omega_3^3 = \lambda_1\omega_1^3 + \lambda_2\omega_2^4.$$

If we denote the coordinates of the current line of the other series of rectilinear generators of the surface of Lie by q^{ij}, then from the conditions of involution [3] we obtain

$$q^{34} = 0, \qquad q^{14} - \lambda q^{23} = 0, \tag{3}$$

$$2\lambda q^{12} + \lambda(\lambda_1 + \lambda\lambda_2)q^{23} + (\lambda^2\alpha' - 2\lambda\beta' - \gamma')q^{42} + (-\lambda^2\gamma - 2\lambda\beta + \alpha)q^{13} = 0.$$

Let the line q be determined by the points $M(x^1x^2x^3x^4)$ and $N(y^1y^2y^3y^4)$; then $q^{ij} = x^iy^j - x^jy^i$. Eliminating q^{ij} and y^i from these and the preceding equations, we obtain the point equation of the quadric of Lie for the ruled surface

$$2\lambda(x^1x^4 - \lambda x^2x^3) - \lambda(\lambda_1 + \lambda\lambda_2)x^3x^4 +$$
$$+(\lambda^2\alpha' - 2\lambda\beta' - \gamma')x^4x^4 + \lambda(\lambda^2\gamma + 2\lambda\beta - \alpha)x^3x^3 = 0. \tag{4}$$

4. The focal plane $A_1A_2A_3$ is tangent to the surface of Lie and, consequently, intersects it in two generators, one of which coincides with the ray of the congruence, while the other is determined by the equations

$$-2\lambda x^2 + (\lambda^2\gamma + 2\lambda\beta - \alpha)x^3 = 0, \qquad x^4 = 0. \tag{5}$$

This generator coincides with the second tangent of the focal net of the surface (A_1) if and only if $\lambda = \pm\sqrt{\alpha/\gamma}$, i.e., only for the ruled surfaces of the congruence determined by the equation $\alpha(\omega_1^3)^2 - \gamma(\omega_2^4)^2 = 0$.

In exactly the same way one can prove that only for the ruled surfaces $\alpha'(\omega_2^4)^2 - \gamma'(\omega_1^3)^2 = 0$ do the two generators of the quadric coincide with the two tangents of the focal net of the surface (A_2) at the point A_2.

The quantities

$$F = \frac{\alpha(\omega_1^3)^2 - \gamma(\omega_2^4)^2}{\alpha(\omega_1^3)^2 + \gamma(\omega_2^4)^2}, \qquad F' = \frac{\gamma'(\omega_1^3)^2 - \alpha'(\omega_2^4)^2}{\gamma'(\omega_1^3)^2 + \alpha'(\omega_2^4)^2}$$

are absolute invariants, whose geometric meaning may serve as the subject of investigation. The equality $F = F'$ characterizes the congruence W, and $FF' = 1$ characterizes the congruence V.

5. It is known that to each ruled surface of a congruence there corresponds a certain line on the focal surface, along which these two surfaces are tangent to each other.

The generator (5) of the quadric of the ruled surface $\omega_2^4 = \lambda\omega_1^3$ coincides with the tangent of the corresponding line ($\omega_2^4 = \lambda\omega_1^3$) if and only if $\lambda = \pm\sqrt{-\alpha/\gamma}$. Consequently, only for the ruled surfaces $\alpha(\omega_1^3)^2 + \gamma(\omega_2^4)^2 = 0$ does the second generator of the quadric of Lie coincide with the tangents to the lines corresponding to these ruled surfaces (i.e., with the asymptotic tangents).

As is known, the generators of the Lie quadric of any nondevelopable surface coincide with the asymptotic tangents ([1]). From this and from the theorems stated above there also follows another theorem, namely:

The line of contact of the focal and ruled surfaces of a congruence can be asymptotic for the ruled surface if and only if this line is an asymptotic line for the focal surface.

6. Take two ruled surfaces of the congruence $\omega_2^4 = \lambda\omega_1^3$ and $\omega_2^4 = \mu\omega_1^3$, and require that the Lie quadrics of these ruled surfaces have a common generator (apart from the ray of the congruence A_1A_2).

The line determined by the points $M = a_1M_1 + a_2M_2 + a_3M_3$, $N = a_1'M_1 + a_2'M_2 + a_4'M_4$ will lie on these two quadrics if and only if the equations

$$
\begin{aligned}
&2\lambda a_2 - (\lambda^2\gamma + 2\lambda\beta - \alpha)a_3 = 0, && 2\mu a_2 - (\mu^2\gamma + 2\mu\beta - \alpha)a_3 = 0,\\
&2\lambda a_1' + (\lambda^2\alpha' - 2\lambda\beta' - \gamma')a_4' = 0, && 2\mu a_1' + (\mu^2\alpha' - 2\mu\beta' - \gamma')a_4' = 0,\\
&2a_1 - 2\lambda a_2 - \lambda_1 - \lambda\lambda_2 = 0, && 2a_1 - 2\mu a_2 - \mu_1 - \mu\mu_2 = 0.
\end{aligned}
$$

are satisfied.

From the first four equations we obtain

$$\lambda\mu\gamma + \alpha = 0, \qquad \alpha\alpha' - \gamma\gamma' = 0. \tag{6}$$

The remaining equations determine the position of the points M and N. The second equation (6) shows that the congruence (A_1A_2) is a congruence W.

Thus, *two Lie quadrics of two ruled surfaces of a congruence have a common generator point if and only if this congruence is a congruence W.*

In a congruence W, one of such ruled surfaces is prescribed arbitrarily, while the second surface is determined by the first equation (6).

7. Through each ray of the congruence there pass ∞^1 ruled surfaces and, consequently, the same number of Lie quadrics. Let us see in what case all the demiquadrics (one series of rectilinear generators of these Lie quadrics) belong to one linear complex.

If the coordinates of this complex are denoted by q^{ij}, then, according to the requirement, equations (3) must be satisfied for any λ, i.e., we must have

$$q^{34} = q^{14} = q^{23} = 0, \qquad q^{12} - \beta' q^{42} - \beta q^{13} = 0,$$

$$\alpha' q^{42} - \gamma q^{13} = 0, \qquad \gamma' q^{42} - \alpha q^{13} = 0.$$

From these equations (if the linear complex does not coincide with the ray of the congruence) we shall have $\alpha\alpha' - \gamma\gamma' = 0$, and the required linear complex is written in the form

$$a = (\beta'\alpha + \beta\gamma')[12] + \gamma'[13] + \alpha[42].$$

Consequently, *all the demiquadrics associated with the ray of a congruence belong to one linear complex if and only if the congruence is a congruence W.* It is known that the complex a is the osculating complex of the congruence W, and, consequently, *all the demiquadrics associated with the ray of a congruence W belong to the osculating linear complex.*

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REFERENCES

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[2] S. P. Finikov, *Cartan' s Method of Exterior Forms*, 1947.

[3] S. P. Finikov, *Theory of Congruences*, Moscow–Leningrad, 1950.

Note: Figure translations are in progress. See original paper for figures.