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Abstract

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MATHEMATICS

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ON LINEAR EXTRAPOLATION OF A DISCRETE HOMOGENEOUS RANDOM FIELD

(Presented by Academician A. N. Kolmogorov on 20 VIII 1956)

Definition. A **discrete homogeneous random field** is a family of complex random variables $x(s,t)$, where s,t are integers and $-\infty < s < \infty$, $-\infty < t < \infty$, such that $M|x(s,t)|^2 < \infty$ and

$$B_x(s,t) = M[x(s+m,t+n)\overline{x(s+t)}] \tag{1}$$

does not depend on m and n.

The function $B_x(s,t)$ is, obviously, positive definite and therefore (see, for example, ([1])):

$$B_x(s,t) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(s\lambda+t\mu)}\, dF_x(\lambda,\mu), \tag{2}$$

where $F_x(\lambda,\mu)$ is an unnormalized two-dimensional distribution function, called the **spectral function** of the field $\{x(s,t)\}$.

Let H_x be the minimal closed linear subspace containing all the variables $x(s,t)$, and let $H_x(t)$ be the minimal closed linear subspace containing all $x(m,n)$ with $-\infty < m < \infty$ and $n \leq t$. Denote

$$S_x = \bigcap_t H_x(t). \tag{3}$$

If

$$S_x = H_x, \tag{4}$$

then the homogeneous random field $\{x(s,t)\}$ will be called **singular**.

Every element $x(s,t)$ of the field $\{x(s,t)\}$ is uniquely represented in the form of the sum

$$x(s,t) = \eta(s,t) + \xi(s,t), \tag{5}$$

where $\xi(s,t) \in H_x(0)$, and $\eta(s,t)$ is orthogonal to $H_x(0)$.

Put

$$\rho(s,t) = \|\eta(s,t)\|^2. \tag{6}$$

Obviously,

$$\rho(s,t) = \rho(s',t) = \rho(t),$$

$$\rho(t_1) \leq \rho(t_2) \quad \text{for } t_1 \leq t_2, \tag{7}$$

so that there exists

$$\lim_{t \to +\infty} \rho(t) = \sigma_\infty^2. \tag{8}$$

If

$$\sigma_\infty^2 = M|x(s,t)|^2 = \|x\|^2, \tag{9}$$

then the homogeneous random field $\{x(s,t)\}$ will be called **regular** (cf. ([1–3])).

It is not difficult to see that, in order for the homogeneous random field $\{x(s,t)\}$ to be regular, it is necessary and sufficient that the equality $S_x = 0$ hold.

We shall indicate the conditions imposed on the spectral function which guarantee the regularity or singularity of the field $\{x(s,t)\}$.

Theorem 1. In order that the homogeneous random field $\{x(s,t)\}$ be regular, it is necessary and sufficient that the following conditions be satisfied:

a) the measures $dF_x(\lambda,\mu)$ and $dF_x(\lambda,\pi)\,d\mu$ (i.e., the product of the measure $dF_x(\lambda,\pi)$ by the measure $d\mu$) be absolutely continuous with respect to each other on the square $-\pi \leq \lambda \leq \pi,\ -\pi \leq \mu \leq \pi$;

b) for almost all values of λ (with respect to the measure $dF_x(\lambda,\pi)$),

$$\left| \int_{-\pi}^{\pi} \log \left(\frac{dF_x(\lambda,\mu)}{dF_x(\lambda,\pi)\,d\mu} \right) d\mu \right| < +\infty. \tag{10}$$

Theorem 2. In order that the homogeneous random field $\{x(s,t)\}$ be regular, it is necessary and sufficient that its spectral function can be represented in the form

$$F_x(\lambda,\mu) = \int_{-\pi}^{\lambda}\int_{-\pi}^{\mu} |L(\lambda,\mu)|^2 \, dG(\lambda)\, d\mu, \tag{11}$$

where $G(\lambda)$ is some function, nondecreasing on the interval $[-\pi, \pi]$ and such that $G(\pi) - G(-\pi) > 0$, and $L(\lambda, \mu)$ is a complex-valued function, different from zero almost everywhere with respect to $dG(\lambda)\, d\mu$, representable in the form

$$L(\lambda,\mu) = \sum_{n=0}^{+\infty} l_n(\lambda) e^{-in\mu} \qquad (l_n(\lambda) \in L^2(dG(\lambda)),\ n = 0, 1, 2, \ldots).$$

Theorems 1 and 2 give two forms of the necessary and sufficient condition for regularity. Let us note that the spectral function of a regular homogeneous random field need not be absolutely continuous, as the following simple example shows:

$$x(s,t) \equiv x(t),$$

where $x(t)$ is a regular stationary random sequence.

Theorem 3. In order that the spectral function of a regular homogeneous random field $\{x(s,t)\}$ be absolutely continuous, it is necessary and sufficient that $x(s,t)$ can be represented in the form

$$x(s,t) = \sum_{n=0}^{+\infty}\sum_{m=-\infty}^{+\infty} a_{mn} u(s-m, t-n), \tag{12}$$

where $\{u(s,t)\}$ is a family of pairwise uncorrelated random variables with constant variance.

Theorem 4. In order that the homogeneous random field $\{x(s,t)\}$ be singular, it is necessary and sufficient that on the interval $-\pi \le \lambda \le \pi$ the condition

$$\int_{-\pi}^{\pi} \log\left(\frac{dF_x(\lambda,\mu)}{dF_x(\lambda,\pi)\, d\mu}\right) d\mu = -\infty \tag{13}$$

hold almost everywhere (with respect to the measure $dF_x(\lambda, \pi)$), where

$$\frac{dF_x(\lambda,\mu)}{dF_x(\lambda,\pi)\, d\mu}$$

is the absolutely continuous part of the measure $dF_x(\lambda,\mu)$ with respect to the measure $dF_x(\lambda,\pi)\, d\mu$.

We shall assume that $Mx(s,t) = 0$. Let

$$x(s, m) = \beta(s, m) + \gamma(s, m), \tag{14}$$

where $\gamma(s, m) = \operatorname{proj}_{H_x(-1)} x(s, m)$, and $\beta(s, m)$ is orthogonal to $H_x(-1)$. Denote

$$\sigma_m^2 = \|\beta(s, m)\|^2. \tag{15}$$

(It is clear that this quantity does not depend on s.) Then σ_m will be the mean-square error of linear extrapolation of the field $\{x(s, t)\}$ $m+1$ steps ahead with respect to the variable t. In applications such a problem naturally arises in cases where t plays the role of a time variable, and s of a spatial one.

Theorem 5.

$$\sigma_m^2 = 2\pi \int_{\eta_\lambda} \sum_{k=0}^{m} |\varphi_k(\lambda)|^2 \, dF_x(\lambda, \pi), \tag{16}$$

where

$$\eta_\lambda = \left\{ \lambda; \left| \int_{-\pi}^{\pi} \log\left(\frac{dF_x(\lambda, \mu)}{dF_x(\lambda, \pi)\, d\mu} \right) d\mu \right| < +\infty \right\}, \tag{17}$$

and the functions $\varphi_k(\lambda)$, $(\lambda \in \eta_\lambda)$, are determined from the relations

$$\exp\left[\frac{1}{2} A_0(\lambda) + \sum_{k=1}^{+\infty} A_k(\lambda) \zeta^k \right] = \varphi_0(\lambda) + \varphi_1(\lambda)\zeta + \varphi_2(\lambda)\zeta^2 + \cdots; \tag{18}$$

$$A_k(\lambda) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\mu} \log\left(\frac{dF_x(\lambda, \mu)}{dF_x(\lambda, \pi)\, d\mu} \right) d\mu. \tag{19}$$

In particular,

$$\sigma_0^2 = 2\pi \int_{-\pi}^{\pi} \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left(\frac{dF_x(\lambda, \mu)}{dF_x(\lambda, \pi)\, d\mu} \right) d\mu \right\} dF_x(\lambda, \pi). \tag{20}$$

Formulas (16), (20) are analogous to Kolmogorov' s formulas $(2, 3)$ for the mean square of the error of extrapolation of stationary random sequences.

Definition. A homogeneous random field $\{x(s, t)\}$ is called a **field of Markov type** if $\operatorname{proj}_{H_x(t-1)} x(s, t)$ belongs to the closed linear manifold spanned by the quantities $x(s, t-1)$, $-\infty < s < +\infty$.

Theorem 6. I. In order that a homogeneous random field $\{x(s, t)\}$ be a field of Markov type, it is sufficient that the condition

$$F_x(\omega) = \iint_\omega \frac{1}{|1 - l(\lambda)e^{-i\mu}|^2}\, dG(\lambda)\, d\mu \quad \text{for all } \omega \subseteq \Omega, \tag{21}$$

be satisfied, where:

a) $G(\lambda)$ is some real nondecreasing bounded function on the interval $-\pi \leq \lambda \leq \pi$, with $G(\pi) - G(-\pi) > 0$;

b) $l(\lambda)$ is a complex function such that $|l(\lambda)| \leq 1$ and

$$\int_{-\pi}^{\pi} \frac{dG(\lambda)}{1 - |l(\lambda)|^2} < +\infty; \tag{22}$$

c)

$$\Omega = \{(\lambda, \mu);\ 1 - l(\lambda)e^{-i\mu} \neq 0\}. \tag{23}$$

II. If the field $\{x(s,t)\}$ is a field of Markov type, then there exist functions $l(\lambda)$ and $q(\lambda)$ such that

$$F_x(\omega) = \iint_\omega \frac{q(\lambda)}{|1 - l(\lambda)e^{-i\mu}|^2}\, dF_x(\lambda, \pi)\, d\mu \tag{24}$$

for all $\omega \subseteq \Omega$, where

$$\text{a)} \quad \Omega = \{(\lambda, \mu);\ 1 - l(\lambda)e^{-i\mu} \neq 0\};$$

$$\text{b)} \quad |l(\lambda)| \leq 1 \qquad (-\pi \leq \lambda \leq \pi);$$

$$\text{c)} \quad q(\lambda) \geq 0 \qquad (-\pi \leq \lambda \leq \pi).$$

III. The field $\{x(s,t)\}$ is singular if and only if $q(\lambda) \equiv 0$, and is regular if and only if $F_x(\overline{\Omega}) = 0$, where $\overline{\Omega}$ is the complement of Ω.

Theorem 7. If the homogeneous random field $\{x(s,t)\}$ is a regular field of Markov type, then

$$\operatorname{proj}_{H_{x(t-1)}} x(s,t) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i[s\lambda + (t-1)\mu]} l(\lambda)\, dr_x(\lambda, \mu), \tag{25}$$

where

$$l(\lambda) = \int_{-\pi}^{\pi} e^{i\mu} \frac{dF_x(\lambda, \mu)}{dF_x(\lambda, \pi)\, d\mu}\, d\mu.$$

The proofs of Theorems 1-3 and 5 are carried out analogously to the proofs of the corresponding theorems for the one-dimensional case (see ([2],[3]), and also ([4–6])).

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