

# ON THE “SECOND” FORM OF MOTION OF AN IDEAL FLUID AROUND A STREAMLINED BODY

1957

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-195701.87717>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Fig. 1

Figure 1: Fig. 1

**Abstract****Full Text****HYDROMECHANICS****A. A. NIKOLSKII****ON THE “SECOND” FORM OF MOTION OF AN IDEAL FLUID AROUND A STREAM-LINED BODY****(A STUDY OF SEPARATED VORTEX FLOWS)***(Presented by Academician A. A. Dorodnitsyn, 23 IV 1957)*

In the work of L. Prandtl <sup>(1)</sup>, the idea was put forward that it is expedient to consider a second possibility for the flow of an ideal fluid past the sharp corners of a body—the formation of a separated flow (in particular, a model flow around a wedge), with the inception and development, near the body, of a vortex spiral surface. The theory of this special form of motion, which in what follows the author of the present paper calls the “second” form of motion, has not yet received sufficient development.

When the second form of motion is realized, in the region of reduced pressure a fixed mass of fluid captures surrounding fluid particles, and these gradually accumulate between the coils of an expanding spiral surface  $\gamma$ , consisting of fixed particles (Fig. 1).

**Fig. 1**

The formation of the second form of motion in the flow of an ideal fluid around bodies reduces to the appearance, within the fluid, of surfaces  $\gamma$  of tangential discontinuity of velocity. To each such flow  $E$ , with velocity vector  $\mathbf{V}(x, y, z, t)$ , one can uniquely associate a basic continuous (without surfaces of velocity discontinuity) potential flow  $E_1$ , with velocity vector  $\mathbf{V}_1(x, y, z, t)$ , satisfying the same boundary conditions and conditions at infinity as the field  $E$ . The velocity field of the flow  $E$  at any  $t$  is the sum of two fields: the field  $E_1$  and an additional field  $E_2$ , with velocity vector  $\mathbf{V}_2(x, y, z, t) = \mathbf{V} - \mathbf{V}_1$ . The field  $E_2$  is potential everywhere outside  $\gamma$  and satisfies the conditions: equality to zero of the normal component of velocity on the surface  $S$  of the body; boundedness of the velocity potential at infinity; and the presence in it of surfaces  $\gamma$  having the same configuration and the same distribution of circulation as in the general flow  $E$ .

Let us consider several cases of plane flows of the simplest bodies by an ideal fluid, with formation of the second form of motion. We shall denote:  $w$  – the complex potential;  $w_1$  – the complex potential of the basic flow  $E_1$ ;  $\gamma$  – the vortex sheet;  $\Gamma$  – the total circulation (counterclockwise) of the piece of  $\gamma$  cut off by the point of it under consideration;  $x, y$  – a rectangular coordinate system;  $z = x + iy$ ;  $t$  – time;  $\rho$  – density;  $L$  – dimension of length;  $T$  – dimension of time;  $\bar{a}$  – the quantity complex-conjugate to  $a$ ;  $|a|$  – the modulus of  $a$ ;  $[a]$  – the dimension of  $a$ . The flows are considered for  $t > 0$ , with  $w = w_1$  at  $t = 0$ . The condition is imposed that there be no force action on the elements of  $\gamma$ . Integrals along  $\gamma$  are understood in the sense of Cauchy. The principles of the theory of similarity are used.

(see, for example, (2)).  $\gamma$  in the general case may consist of separate pieces  $\gamma_q$  ( $q = 1, \dots, N$ ). Let  $z = z_q(\Gamma, t)$  be the equation of the  $q$ -th piece of  $\gamma$ .

I.  $w_1 = f_1(t)z^n + f_2(t)z^{2n}$ ;  $n$  is a real constant  $1/2 \leq n < 1$ ;  $f_1(t)$ ,  $f_2(t)$  are real functions.  $w_1$  gives the flow around a wedge with opening

$$\frac{2n-1}{n} \pi.$$

The functions  $z_q(\Gamma, t)$  must satisfy the system of equations

$$|\zeta_p(\Gamma_0, t)|^{2(1-n)/n} \frac{\partial \overline{\zeta_p(\Gamma_0, t)}}{\partial t} = n^2 \left\{ f_1(t) + 2f_2(t)\zeta_p(\Gamma_0, t) - \frac{1}{2\pi i} \sum_{q=1}^N \int_{\gamma_q} \frac{d\Gamma}{\zeta_p(\Gamma_0, t) - \zeta_q(\Gamma, t)} + \frac{1}{2\pi i} \sum_{q=1}^N \int_{\gamma_q} \frac{d\Gamma}{\zeta_p(\Gamma_0, t) - \overline{\zeta_q(\Gamma, t)}} \right\}. \quad (1)$$

Here  $\zeta(\Gamma, t) = \{z(\Gamma, t)\}^n$ ;  $\Gamma_0$  is the value of  $\Gamma$  at an arbitrary point of the curve  $\gamma_p$ ; integration is carried out from points lying on the wall.

- a)  $f_1(t) = k_1 t^m$ ;  $f_2(t) = k_2 t^{[2(1-n)m-n]/(2-n)}$ ,  $k_1 = \text{const}$ ,  $k_2 = \text{const}$ ,  $m$  is a real constant. The flow may have physical meaning for  $m > -1$ , when the path integral of particles converges as  $t \rightarrow 0$ .  $[k_1] = L^{2-n} T^{-(m+1)}$ ,  $[k_2] = L^{2(1-n)} T^{2(m+1)(n-1)/(2-n)}$ . The flow may be assumed self-similar. The system of equations (1), under the assumption that  $\gamma$  consists of a single piece, reduces to the following equation in dimensionless quantities:

$$|\mu(G_0)|^{2(1-n)/n} \left\{ n(m+1) \overline{\mu(G_0)} - (2m+n) G_0 \frac{d\mu(G_0)}{dG_0} \right\} = n^2(2-n) \left\{ 1 + 2\beta \mu(G_0) - \frac{1}{2\pi i} \int_{\gamma} \frac{dG}{\mu(G_0) - \mu(G)} + \frac{1}{2\pi i} \int_{\gamma} \frac{dG}{\mu(G_0) - \overline{\mu(G)}} \right\}. \quad (2)$$

Throughout the paper it is assumed that

$$\mu = Re^{i\vartheta} = r^n e^{in\theta} = k_1^{n/(n-2)} t^{n(m+1)/(n-2)} \xi = k_1^{n/(n-2)} z^n, \quad \nu = \xi + i\eta = \mu^{1/n},$$

$$G = k_1^{2/(n-2)} t^{(2m+n)/(n-2)} \Gamma; \quad \beta = k_1^{2(n-1)/(2-n)} k_2; \quad (3)$$

$\beta$  is a constant. Formulas (3) give the law of expansion of  $\gamma$  and of the change of its circulation in time.  $\Gamma$  at corresponding points increases with time if  $2m + n > 0$ , decreases if  $2m + n < 0$ , and remains unchanged if  $2m + n = 0$ .

- b) A subcase of item a), when  $2m + n = 0$ . Analysis of equation (2) shows that as  $2m + n \rightarrow 0$ ,  $dG_0/d\mu(G_0) \rightarrow 0$  on all turns of  $\gamma$ . Gradually an ever larger number of the outer turns of  $\gamma$  is freed from vortices, while the circulation of the vortices is concentrated on the inner turns of  $\gamma$ ; for  $2m + n = 0$  all of  $\gamma$  is free of vortices, and at its center there is a discrete vortex. Here the total circulation does not change with time, and, indeed, one may seek a solution in which the entire system of vortices reduces to a discrete vortex at the point  $z = z_1(t)$  with circulation  $-\Gamma_1$  unchanged in time. Put  $\mu_1 = k_1^{n/(n-2)} t^{-n/2} z_1^n$ ,  $G_1 = k_1^{2/(n-2)} \Gamma_1$ ;  $\mu_1$  is a complex constant;  $G_1$  is a real constant. The condition that there be no action of an external force at  $z = z_1(t)$  gives the equation

$$\frac{1}{2} |\mu_1|^{2/n} = n\mu_1 + 2n\beta\mu_1^2 - \frac{G_1}{2\pi i} \left( \frac{n-1}{2} - n \frac{\mu_1}{\mu_1 - \bar{\mu}_1} \right). \quad (4)$$

Putting  $\mu_1 = R_1 e^{i\vartheta_1}$ , we obtain

$$\frac{1}{2} R_1^{2/n} = nR_1 \cos \vartheta_1 + 2n\beta R_1^2 \cos 2\vartheta_1 - \frac{G_1}{4\pi} n \operatorname{ctg} \vartheta_1;$$

$$nR_1 \sin \vartheta_1 + 2n\beta R_1^2 \sin 2\vartheta_1 = \frac{1}{4\pi} G_1. \quad (5)$$

These two equations relate the three quantities  $R_1, \vartheta_1, G_1$ , and so far the problem has an infinite set of solutions. We shall require that the velocity be finite at the point  $z = 0$ . This condition gives the equation

$$\pi R_1 = G_1 \sin \vartheta_1. \quad (6)$$

In what follows we shall require the conditions of finiteness of velocity at the vertices of the angles in all cases.

Assuming that the whole circulation  $\gamma$  for  $2m + n = 0$  is concentrated at a single point, we obtain, for an arbitrary point  $\mu = \mu_0$  of the spiral  $\gamma$ ,

$$\int_{\gamma} \frac{dG}{\mu_0 - \mu(G)} = \frac{G_1}{\mu_0 - \mu_1}, \quad \int_{\gamma} \frac{dG}{\bar{\mu}_0 - \bar{\mu}(G)} = \frac{G_1}{\bar{\mu}_0 - \bar{\mu}_1},$$

where  $\mu_1$  and  $G_1$  are the same as in formulas (4), (5), (6). Eliminating from equation (2) and its conjugate the quantity  $dG_0/G_0$ , and taking into account the limiting equalities for the integrals given above, we obtain the differential equation for  $\gamma$

$$\frac{d \ln \bar{\mu}}{d \ln \mu} = \frac{\frac{m+1}{2-n} - n\mu|\mu|^{-2/n} \left(1 + 2\beta\mu - \frac{1}{2\pi i} \frac{G_1}{\mu - \mu_1} + \frac{1}{2\pi i} \frac{G_1}{\mu - \bar{\mu}_1}\right)}{\frac{m+1}{2-n} - n\bar{\mu}|\mu|^{-2/n} \left(1 + 2\beta\bar{\mu} + \frac{1}{2\pi i} \frac{G_1}{\bar{\mu} - \mu_1} - \frac{1}{2\pi i} \frac{G_1}{\bar{\mu} - \bar{\mu}_1}\right)}. \quad (7)$$

In the planes  $\mu$  and  $v$ , the image of a fixed particle moving in the  $z$ -plane moves along the “trajectory” of automodel motion. The differential equation of the trajectories has the form

$$\frac{d \ln \bar{\mu}}{d \ln \mu} = \frac{\frac{m+1}{2-n} - n\mu|\mu|^{-2/n} \frac{dW}{d\mu}}{\frac{m+1}{2-n} - n\bar{\mu}|\mu|^{-2/n} \frac{d\bar{W}}{d\bar{\mu}}} \quad (W = k_1^{2/(n-2)} t^{(2m+n)/(n-2)} w). \quad (8)$$

The equations of the trajectories do not depend on  $t$ . The curves of the  $z$ -plane corresponding to the trajectories, for any  $t$ , consist of the same particles. In automodel motion,  $\gamma$  satisfies equation (8) when approached from both sides, being the line of contact of two different trajectories. Equation (7) is a special form of equation (8) for the vortex solution. To determine  $\gamma$  in this case it is necessary to find the integral curve of equation (7) which leaves the body and winds onto the vortex point.

For  $\beta = 0$  the system of equations gives

$$R_1 = \left\{ (1-n)\sqrt{n}\sqrt{4n-1} \right\}^{n/(2-n)}; \quad G_1 = 2\pi\sqrt{n} R_1; \quad \sin \vartheta_1 = \frac{1}{2\sqrt{n}}. \quad (9)$$

II.  $\omega_1 = f_1(t)\sqrt{z} + f_2(t)z$ . This subcase of item I corresponds to the development of the second form of motion near the half-plane  $y = 0$ ,  $x \geq 0$ . For a flow with potential  $\omega_1$ , at the point  $z = 0$  a concentrated (suction) force

$$X = -\frac{\pi}{4}\rho f_1(t)^2$$

is applied to the fluid. Application of the theorem on the quantity of motion in projection onto the  $x$ -axis (under the condition that the velocity is finite at

Fig. 2

Figure 2: Fig. 2

Fig. 3

Figure 3: Fig. 3

$z = 0$ ) shows that the absence of the force  $X$  is compensated by an increment of the quantity of motion along the  $x$ -axis of the additional field  $E_2$ . The following integral theorem is valid:

$$\frac{d}{dt} \int_{\gamma} y d\Gamma = \frac{\pi}{4} f_1(t)^2. \quad (10)$$

Integration is carried out over all parts of  $\gamma$ , beginning from the points on the wall.

- a)  $f_1(t) = k_1 t^m$ ,  $f_2(t) = k_2 t^{(2m-1)/3}$ . This is the subcase of item Ia) ( $n = 1/2$ ). Theorem (10), in dimensionless quantities, has the form

$$(2m + 1) \int_{\gamma} R^2 \sin 2\vartheta dG = (2m + 1) \int_{\gamma} r \sin \theta dG = \frac{\pi}{4} \quad (m > -1/2). \quad (11)$$

- b)  $f_1(t) = k_1 t^{-1/4}$ ,  $f_2(t) = k_2 t^{-1/2}$ . This is the subcase of item 1b) ( $n = 1/2$ ). The solution of the system of equations (5), (6) gives

$$R_1 = \frac{1}{\sqrt[3]{4 \cos \vartheta_1}}; \quad G_1 = \frac{\pi}{\sqrt[3]{4 \cos \vartheta_1 \sin \vartheta_1}}; \quad \beta = \frac{\cos 2\vartheta_1 \sqrt[3]{4 \cos \vartheta_1}}{4 \sin \vartheta_1 \cos 2\vartheta_1}. \quad (12)$$

Relation (11) is expressed as the equality  $R_1^2 \sin 2\vartheta_1 G_1 = \pi/2$ , which solution (12) satisfies. The point  $z = 0$  is a branching point of the flow.

**Fig. 2**

**Fig. 3**

The sign of  $d\omega/dz$  at the point  $z = 0$  coincides with the sign of the quantity  $\beta R_1 - \cos \vartheta_1$ . Denote by  $\beta_{cr}$  the value of  $\beta$  when  $\beta R_1 = \cos \vartheta_1$ . For  $\beta < \beta_{cr}$ , at the point  $z = 0$ ,  $d\omega/dz < 0$ . The point  $z = 0$  is the point of "separation" of the flow, i.e. the point at which the trajectory  $\gamma$  leaves the body (Fig. 2). For  $\beta > \beta_{cr}$ , at the point  $z = 0$ ,  $d\omega/dz > 0$ , and it is not the point at which  $\gamma$  leaves the body. The point  $C$  of separation of the flow is located at  $x > 0$  (Fig. 3). For  $\beta \rightarrow \infty$  we have  $\vartheta_1 \rightarrow 0$ ,  $G_1 \rightarrow \infty$ ,  $R_1 \rightarrow 1/\sqrt[3]{4}$ . In the  $\mu$ -plane, as  $\beta \rightarrow \infty$ ,

Fig. 4

Figure 4: Fig. 4

the spiral  $\gamma$  is localized near the point  $\mu = 1/\sqrt[3]{4}$ , contracting into this point. In the  $z$ -plane,  $\gamma$  is localized near the point  $y = 0$ ,  $x = 4^{-2/3}k_1^{2/3}\sqrt{t}$  (Fig. 4).

**Fig. 4**

- c)  $w_1 = k_1 t^m \sqrt{z} + k_2 t^{(2m-1)/3} z$ ;  $\beta = k_1^{-2/3} k_2 \gg 1$ —the subcase of item a). Let us assume, by analogy with item b), that as  $\beta \rightarrow \infty$  the whole of  $\gamma$  contracts to the point  $\mu = R_1$  of the real axis. The condition of finite velocity at the point  $z = 0$  and theorem (11) give, respectively,

$$\int_{\gamma} \frac{1}{R} \sin \vartheta dG = \pi; \quad \int_{\gamma} R^2 \sin 2\vartheta dG = \frac{\pi}{4} \frac{1}{2m+1}. \quad (13)$$

Putting  $\vartheta \rightarrow 0$ ,  $R \rightarrow R_1$ , we obtain the equalities

$$R_1 = \frac{1}{2}(2m+1)^{-1/3}, \quad \int_{\gamma} \vartheta dG = \frac{\pi}{2}(2m+1)^{-1/3}. \quad (14)$$

The principal results of the work were reported by the author at the IX International Congress on Applied Mechanics in Brussels, 12 IX 1956.

Institute of Mechanics  
Academy of Sciences of the USSR

Received  
9 XI 1956

**REFERENCES**

1. L. Prandtl, Über die Entstehung von Wirbeln in der idealen Flüssigkeit, Vorträge aus Hydro- und Aerodynamik, Berlin, 1924.
2. L. I. Sedov, *Methods of Similarity and Dimension in Mechanics*, Moscow, 1954.

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*