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# MATHEMATICS

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## Abstract

## Full Text

MATHEMATICS

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# ON THE EXISTENCE AND UNIQUENESS OF THE SOLUTION OF AN EQUATION OF GENERAL FORM IN CONNECTION WITH INVERTIBILITY IN SEMIGROUPS OF TRANSFORMATIONS

*(Presented by Academician V. I. Smirnov, 15 IV 1957)*

1°. A set considered with respect to a certain single-valued associative operation of multiplication defined on it is called a **semigroup**. Without giving proofs, let us consider some general properties of semigroups, clarify their role for semigroups of transformations, and connect this with the question of solutions of an equation of general form.

2°. Let  $S$  be an element of a semigroup  $\mathfrak{A}$ . If for every  $A \in \mathfrak{A}$  there is in  $\mathfrak{A}$  such an element  $X$  that  $XS = A$ , then  $S$  is said to be **invertible on the left**. **Invertibility on the right** is defined analogously. An element invertible both on the left and on the right is called **two-sided invertible**.

For a semigroup  $\mathfrak{A}$  we shall denote by  $\mathfrak{A}_1$  the set of its two-sided invertible elements; by  $\mathfrak{A}_l$  the set of elements invertible on the left but not invertible on the right; by  $\mathfrak{A}_r$  the set of elements invertible on the right but not invertible on the left; and by  $\mathfrak{A}_0$  the set of elements invertible neither on the left nor on the right. It can be proved that each of the four indicated subsets of  $\mathfrak{A}$  is its subsemigroup. It is known <sup>(1)</sup> that the set  $\mathfrak{A}_1$ , if it is nonempty, is a group, whose identity  $E$  serves as an identity for the whole semigroup  $\mathfrak{A}$  (i.e.,  $AE = EA = A$  for every  $A \in \mathfrak{A}$ ). Thus, semigroups with an identity, and only they, have a nonempty subsemigroup  $\mathfrak{A}_1$ .

3°. **Theorem.** *If  $XY \in \mathfrak{A}_1$  ( $X, Y \in \mathfrak{A}$ ), then one of the following two alternatives holds: either  $X, Y \in \mathfrak{A}_1$ , or  $X \in \mathfrak{A}_r$  and  $Y \in \mathfrak{A}_l$ .*

Of course, the converse conclusion cannot be drawn. In some semigroups (for example, in the semigroup of all transformations of any infinite set) a product of the form  $XY$ , where  $X \in \mathfrak{A}_r$  and  $Y \in \mathfrak{A}_l$ , with a suitable choice of the pair  $(X, Y)$ , may turn out to belong to any of the four sets  $\mathfrak{A}_1, \mathfrak{A}_l, \mathfrak{A}_r, \mathfrak{A}_0$ .

4°. The property of one-sided invertibility of elements is connected with the property of cancellability <sup>(2)</sup>, which, in turn, is connected with various properties of

semigroups <sup>(2,3)</sup>. An element  $S$  of a semigroup  $\mathfrak{A}$  is called **right cancellative** if there exists such an  $\mathfrak{A}' \subset \mathfrak{A}$ ,  $\mathfrak{A}' \neq \mathfrak{A}$ , that  $\mathfrak{A}'S = \mathfrak{A}$ . A left cancellative element is defined analogously.

**Theorem.** *All left cancellative elements of the semigroup  $\mathfrak{A}$  are contained in  $\mathfrak{A}_r$ . All right cancellative elements are contained in  $\mathfrak{A}_l$ .*

Since  $\mathfrak{A}_l \cap \mathfrak{A}_r$  is empty, it follows from this, in particular, the known fact <sup>(2)</sup> that no element can be simultaneously left and right cancellative.

5°. If the subsemigroup  $\mathfrak{A}_l$  of a semigroup  $\mathfrak{A}$  is empty, some elements of  $\mathfrak{A}_r$  may fail to be left cancellative. Analogously for  $\mathfrak{A}_l$ . The situation is different in semigroups possessing an identity.

**Theorem.** In a semigroup with identity,  $\mathfrak{A}_r$  is the set of all its left cancellable elements, and  $\mathfrak{A}_l$  is the set of all its right cancellable elements.

6°. As a consequence of the theorem of §5°, the fulfillment of certain properties of one-sided invertibility entails the two-sided invertibility of an element.

**Corollary.** If in a semigroup  $\mathfrak{A}$  with identity, for an element  $X \in \mathfrak{A}$ , the equality  $X\mathfrak{A}' = \mathfrak{A}$  ( $\mathfrak{A}' \subset \mathfrak{A}$ ) holds if and only if  $\mathfrak{A}' = \mathfrak{A}$ , then  $X$  is two-sided invertible in  $\mathfrak{A}$ .

7°. If  $S$  is contained in some subsemigroup  $\mathfrak{B}$  of the semigroup  $\mathfrak{A}$ , then cases are possible in which  $S$  is left invertible in  $\mathfrak{B}$ , but not left invertible in  $\mathfrak{A}$ , or conversely, left invertible in  $\mathfrak{A}$ , but not left invertible in  $\mathfrak{B}$ .

A subsemigroup  $\mathfrak{B}$  of a semigroup  $\mathfrak{A}$  is called **proper with respect to left invertibility** if every element  $S$  of  $\mathfrak{B}$  is left invertible in  $\mathfrak{B}$  if and only if it is left invertible in  $\mathfrak{A}$ .

Analogously for right invertibility.

8°. A mapping of some set  $\Omega$  into itself will be called a **transformation** of  $\Omega$ . If for transformations  $X, Y, Z$ , for every  $\alpha \in \Omega$ , one has

$$X(Y\alpha) = Z\alpha,$$

then  $Z$  is called the **product** of  $X$  and  $Y$  ( $Z = XY$ ). We denote the set of all transformations of  $\Omega$  by  $\mathfrak{S}_\Omega$ . It is a semigroup. Its subsemigroups are called **semigroups of transformations** of  $\Omega$ . A semigroup of transformations  $\mathfrak{A}$  is called **proper with respect to left invertibility** if  $\mathfrak{A}$  is a subsemigroup of the semigroup  $\mathfrak{S}_\Omega$  proper with respect to left invertibility.

9°. Let  $S$  be a given arbitrary transformation of the set  $\Omega$ . The equation  $S\xi = \alpha$  ( $\alpha \in \Omega$ ), with respect to the unknown  $\xi \in \Omega$ , will be called an **equation of general form**.

Let  $S$  be an element of some semigroup of transformations of  $\Omega$ . Sometimes certain properties of the indicated equation may be connected with certain properties (moreover abstract ones, i.e. invariant under isomorphisms) of  $S$  as an element of this semigroup.

**Theorem.** Let  $\mathfrak{A}$  be a semigroup of transformations of the set  $\Omega$ , proper with respect to right invertibility, and let  $S \in \mathfrak{A}$ . In order that the equation  $S\xi = \alpha$ , with respect to the unknown  $\xi \in \Omega$ , be solvable for every  $\alpha \in \Omega$ , it is necessary and sufficient that  $S$  be a right-invertible element of the semigroup  $\mathfrak{A}$ .

**10°.** Not only the solvability of the equation, but also the question of uniqueness of the solution, belongs to the number of the indicated properties.

**Theorem.** Let  $\mathfrak{A}$  be a semigroup of transformations of the set  $\Omega$ , proper with respect to left invertibility, and let  $S \in \mathfrak{A}$ . In order that the equation  $S\xi = \alpha$ , with respect to the unknown  $\xi \in \Omega$ , have at most one solution for every  $\alpha$ , it is necessary and sufficient that  $S$  be a left-invertible element of the semigroup  $\mathfrak{A}$ .

**11°.** If  $\mathfrak{A}$  is some semigroup of transformations of the set  $\Omega$ , proper both with respect to left invertibility and with respect to right invertibility, then the theorems of §§9° and 10° show what role, for the basic properties of the equation  $S\xi = \alpha$ , is played by the membership of the element  $S \in \mathfrak{A}$  in one or another of the subsemigroups  $\mathfrak{A}_1, \mathfrak{A}_l, \mathfrak{A}_r, \mathfrak{A}_0$ . If  $S \in \mathfrak{A}_1$ , then the equation is solvable for all  $\alpha$  and always has a unique solution. If  $S \in \mathfrak{A}_l$ , then the equation is not solvable for all  $\alpha$ , but in the case of solvability has only a unique solution. If  $S \in \mathfrak{A}_r$ , then the equation is solvable for all  $\alpha$ , but for some of them has several distinct solutions. Finally, for  $S \in \mathfrak{A}_0$ , the equation is not solvable for all  $\alpha$ , and for some  $\alpha$  has several distinct solutions.

If  $\mathfrak{A}$  contains the identity transformation (which, obviously, is the identity of  $\mathfrak{A}$ ), then from the theorem of §5°, by virtue of what has been said, there follows the role of the cancellability properties of elements of  $\mathfrak{A}$  for the indicated properties of the equation of general form. In particular, from the fact that a semigroup with identity,

possessing left augmenting elements necessarily also possesses right augmenting elements, just as conversely<sup>(2)</sup>; a number of obvious corollaries follow. For example, if  $S\xi = \alpha$  is solvable for all  $S \in \mathfrak{A}$  and for all  $\alpha \in \Omega$ , then the solution is always unique. The converse proposition also holds.

**12°.** Let  $\Gamma$  be a certain partially ordered set. Denote by  $\mathfrak{M}_\Gamma$  the totality of all transformations  $S \in \mathfrak{S}_\Gamma$  which do not violate the relation of partial ordering, i.e. such that  $\alpha \leq \beta$  and  $S\alpha \geq S\beta$  ( $\alpha, \beta \in \Gamma$ ) are compatible only when  $\alpha = \beta$ .  $\mathfrak{M}_\Gamma$  is always a semigroup of transformations regular with respect to right invertibility; however, as examples can show, it is not always regular with respect to left invertibility. It can be proved that one sufficient condition for  $\mathfrak{M}_\Gamma$  to be regular with respect to left invertibility is the fulfillment of the following property of existence of separating elements. For any  $\Gamma' \subset \Gamma$  and  $\Gamma'' \subset \Gamma$  such that  $\alpha' \geq \alpha''$ ,  $\alpha' \in \Gamma'$ ,  $\alpha'' \in \Gamma''$  are compatible only in the case  $\alpha' = \alpha''$ , there must exist a "separating element"  $\gamma \in \Gamma$  possessing the properties: 1)  $\alpha' \geq \gamma$ ,  $\alpha' \in \Gamma'$ , is possible only when  $\alpha' = \gamma$ ; 2)  $\alpha'' \leq \gamma$ ,  $\alpha'' \in \Gamma''$ , is possible only when  $\alpha'' = \gamma$ .

In consequence of what was said above, the solvability and uniqueness of the

solution of the equation  $S\xi = \alpha$  ( $\alpha \in \Gamma$ ) with respect to the unknown  $\xi \in \Gamma$ , in the case where the condition on separating elements is fulfilled in  $\Gamma$ , turn out to depend entirely on the right and left invertibility of the element  $S$  in the semigroup  $\mathfrak{M}_\Gamma$ .

13°. The set  $\mathfrak{A}_l \cup \mathfrak{A}_r \cup \mathfrak{A}_0$  is the totality of all elements of the semigroup  $\mathfrak{A}$  which are not two-sided invertible elements. In some cases this set is a subsemigroup, in others it is not. According to item 3°, the former will hold if and only if, for all  $X \in \mathfrak{A}_r$  and  $Y \in \mathfrak{A}_l$ , one has  $XY \in \mathfrak{A}_l$ . In particular, this will be true when  $\mathfrak{A}_r$  or  $\mathfrak{A}_l$  is empty.

As a consequence of the theorem of item 3°, it follows from what has been said that, in the case when  $\mathfrak{A}_l \cup \mathfrak{A}_r \cup \mathfrak{A}_0$  is a subsemigroup of  $\mathfrak{A}$ , it will necessarily also be a two-sided ideal.

If the set of all elements of the semigroup  $\mathfrak{A}$  which are not two-sided invertible elements is a subsemigroup of it distinct from  $\mathfrak{A}$  (the latter, by item 2°, means that  $\mathfrak{A}$  has an identity), then  $\mathfrak{A}$  is called a **semigroup with a separated group part**.

The notion of a semigroup with a separated group part is closely related to Rauter's notion of a supergroup<sup>(4,5)</sup>.

14°. **Theorem.** *In order that a semigroup with identity be a semigroup with a separated group part, it is necessary and sufficient that it can be represented in the form of a disjoint union of two of its subsemigroups, of which one is a group and the other a left or right ideal.*

15°. It can be shown that semigroups with a separated group part include: commutative semigroups with identity, finite semigroups with identity, semigroups with two-sided cancellation possessing an identity (and, in particular, semigroups embeddable in groups).

16°. **Theorem.** *If a subsemigroup  $\mathfrak{B}$  of a semigroup with separated group part  $\mathfrak{A}$  contains the identity of  $\mathfrak{A}$ , then  $\mathfrak{B}$  itself is a semigroup with a separated group part.*

17°. Since the semigroup of all complex square matrices of one order, as is easy to see, is a semigroup with a separated group part, it follows, using the theorem of item 16°, that the following can be proved.

**Corollary.** *Every semigroup with identity, isomorphically representable by matrices, is a semigroup with a separated group part.*

18°. It can be shown that in semigroups with a separated group part there are no augmenting elements. If, however, a semigroup with identity is not a semigroup with a separated group part, then in it

there necessarily exist both left and right enlarging elements. Hence, by item 5°, the following theorem follows:

**Theorem.** Let  $\mathfrak{A}$  be a semigroup with identity. If  $\mathfrak{A}$  is a semigroup with a separable group part, then both sets  $\mathfrak{A}_l$  and  $\mathfrak{A}_r$  are empty (i.e., every element invertible on the left is also invertible on the right, and conversely). If  $\mathfrak{A}$  is not a semigroup with a separable group part, then both sets  $\mathfrak{A}_l$  and  $\mathfrak{A}_r$  are nonempty.

19°. As a consequence of items 9° and 10° from Theorem item 18°, the following properties of an equation of general form follow. Let  $\mathfrak{A}$  be some semigroup of transformations of a set  $\Omega$ , which is regular with respect to left invertibility and regular with respect to right invertibility. Suppose also that  $\mathfrak{A}$  contains the identity transformation. From what was said in item 11°, there follows a connection between the properties of solvability and uniqueness of the solution of the equation of general form  $S\xi = \alpha$  ( $S \in \mathfrak{A}$ ,  $\alpha \in \Omega$ ) with respect to the unknown  $\xi \in \Omega$ .

If  $\mathfrak{A}$  is a semigroup with a separable group part and the equation  $S\xi = \alpha$  is solvable for every  $\alpha \in \Omega$ , then its solution is always unique. On the other hand, if for some  $\alpha = \alpha_1$  the equation  $S\xi = \alpha_1$  is unsolvable, then there exists an  $\alpha^2 \in \Omega$  for which the equation  $S\xi = \alpha_2$  will have more than one solution.

If  $\mathfrak{A}$  is not a semigroup with a separable group part, then in  $\mathfrak{A}$  there exist such  $S_1$  and  $S_2$  that the equation  $S_1\xi = \alpha$  is solvable for all  $\alpha \in \Omega$ , but for some of them has more than one solution; the equation  $S_2\xi = \alpha$  is unsolvable for some  $\alpha \in \Omega$ , but for every  $\alpha \in \Omega$  for which it is solvable, it has only a unique solution.

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*Note: Figure translations are in progress. See original paper for figures.*

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