SYSTEMS OF SINGULAR INTEGRAL EQUATIONS OF CONVOLUTION TYPE

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Abstract

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SYSTEMS OF SINGULAR INTEGRAL EQUATIONS OF CONVOLUTION TYPE

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In papers ([1–3]), on the basis of the theory of boundary-value problems for analytic functions, convolution-type equations with one unknown function were studied. In the present note we consider systems of singular equations of convolution type of an analogous character. We write them in vector form:

$$f(x)+\frac{1}{\sqrt{2\pi}}\int_0^\infty k_1(x-t)f(t)\,dt+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 k_2(x-t)f(t)\,dt=g(x),\qquad -\infty<x<\infty; \tag{A}$$

$$f(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty k_1(x-t)f(t)\,dt=g(x),\qquad x>0;$$

$$f(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty k_2(x-t)f(t)\,dt=g(x),\qquad x<0, \tag{B}$$

where $K_\alpha(x)=\|k_{ij}^\alpha(x)\|$, $\alpha=1,2$, are matrices; $g(x)$, $f(x)$ are vectors of order n.

As the apparatus for investigating the indicated systems, we use the Fourier transform ([3,4]), the theory of functions of matrices ([5,6]), and the theory of boundary-value problems for analytic functions for a system of unknown functions ([7]). We shall denote by a capital letter the transform of a function denoted by the corresponding lowercase letter. By the signs $+$, $-$ we denote functions with the properties: $f_+(x)\equiv 0$ for $x<0$; $f_-(x)\equiv 0$ for $x>0$; their transforms are denoted by F^+ and F^-. We use the following property of the Fourier transform: if $k(x)e^{-yx}\in L(-\infty,\infty)$ or $L^2(-\infty,\infty)$, $a\le y\le b$, then $K(z)$ is a function analytic in the strip $a<\operatorname{Im} z<b$.

§ 1. Suppose that the kernels of the equations $k_{ij}^\alpha(x)\in L(-\infty,\infty)$, and $g_i(x)\in L^2(-\infty,\infty)$. We seek the solutions $f_i(x)$ in the class $L^2(-\infty,\infty)$.

By applying the Fourier transform to system (A), we arrive at the Riemann boundary-value problem for a system of functions with boundary condition:

$$F^+(x) = A(x)F^-(x) + [E + K_1(x)]^{-1}G(x), \qquad -\infty < x < \infty, \tag{1}$$

where

$$A(x) = [E + K_1(x)]^{-1} \cdot [E + K_2(x)].$$

The solution of the latter, in the general case, can be obtained only from a system of Fredholm integral equations. Therefore it is of interest to consider such cases of system (A) when the solution can be obtained effectively (by means of a finite number of linear transformations ([5]) or in quadratures).

1°. In paper ([5]) it was shown that if $A(x)$ is a functionally commutative matrix,* then the canonical matrix of solutions of problem (1) can

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* A functionally commutative matrix is defined by the equality $A(x_1)A(x_2) = A(x_2)A(x_1)$, where x_1, x_2 are any two values of the argument ([6]).

can be obtained by quadratures. It is easy to show that if $A(x)$ is a functionally commutative matrix, then its Fourier transform—the matrix $a(x)$, and the matrices $A(x) \pm E$, $[A(x)]^{-1}$, will also be functionally commutative.

Proceeding from this, one can show that a necessary and sufficient condition for the functional commutativity of the matrix $A(x)$ is the functional commutativity of the matrices

$$[k_2(x) - k_1(x)] + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r_1(x-t)[k_2(t) - k_1(t)]\, dt; \tag{2}$$

$$[k_2(x) - k_1(x)] + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r_2(x-t)[k_2(t) - k_1(t)]\, dt, \tag{3}$$

where the matrix $r_\alpha(x)$ is the solution of the equation

$$r_\alpha(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r_\alpha(x-t)k_\alpha(t)\, dt = -k_\alpha(x), \qquad \alpha = 1, 2.$$

Conditions (2) and (3) are equivalent.

The system of equations (B) is considered analogously.

2°. In the case where

$$k_\alpha(x) = \begin{cases} k'_\alpha(x), & x > 0, \\ k''_\alpha(x), & x < 0, \end{cases} \qquad \alpha = 1, 2, \tag{4}$$

system (A) takes the form

$$f(x)+\frac{1}{\sqrt{2\pi}}\int_0^x k_1'(x-t)f(t)\,dt+\frac{1}{\sqrt{2\pi}}\int_x^\infty k_1''(x-t)f(t)\,dt+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 k_2'(x-t)f(t)\,dt=g(x),\qquad x>0;$$

$$(5)$$

$$f(x)+\frac{1}{\sqrt{2\pi}}\int_0^\infty k_1''(x-t)f(t)\,dt+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x k_2'(x-t)f(t)\,dt+\frac{1}{\sqrt{2\pi}}\int_x^0 k_2''(x-t)f(t)\,dt=g(x),\qquad x<0,$$

and is solvable by quadratures when:

$$\text{a) } k_1''(x)\equiv k_2'(x)\equiv 0;\qquad \text{b) } k_1''(x)\equiv k_2''(x)\equiv 0;\qquad \text{c) } k_2'(x)\equiv k_2''(x)\equiv 0, \tag{6}$$

under the additional condition of the nonsingularity of the matrices

$$\left.\begin{array}{l}\text{a) } E+K_1'(z) \text{ in } D^+,\qquad E+K_2''(z) \text{ in } D^- \text{ in case (6a);}\\ \text{of the nonsingularity and analyticity of the matrices}\\ \text{b) } [E+K_1'(z)]^{-1}\cdot[E+K_2'(z)] \text{ in } D^+ \text{ in case (6b);}\\ \text{c) } [E+K_1''(z)]^{-1}\cdot[E+K_2''(z)] \text{ in } D^- \text{ in case (6c).}\end{array}\right\} \tag{7}$$

If the additional conditions (7) are not satisfied, then the solution of equation (5), in the presence of (6), is obtained by means of a finite number of linear transformations.

The "paired" equation (B), under assumption (4), has the form:

$$\begin{aligned} f(x)+\frac{1}{\sqrt{2\pi}}\int_x^\infty k_1''(x-t)f(t)\,dt+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x k_1'(x-t)f(t)\,dt=g(x),\quad x>0;\\ f(x)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x k_2'(x-t)f(x)\,dt+\frac{1}{\sqrt{2\pi}}\int_x^\infty k_2''(x-t)f(t)\,dt=g(x),\quad x<0,\end{aligned} \tag{8}$$

and is solvable by quadratures when:

$$\text{a) } k_1'(x)\equiv k_2''(x)\equiv 0;\qquad \text{b) } k_1'(x)\equiv k_2'(x)\equiv 0;\qquad \text{c) } k_1''(x)\equiv k_2''(x)\equiv 0, \tag{9}$$

under the additional condition:

of nonsingularity of the matrices

$$\left.\begin{array}{l} \text{a) } E+K_2'(z) \text{ in } D^+, \quad E+K_1''(z) \text{ in } D^- \text{ in case (9a);} \\ \text{of nonsingularity and analyticity of the matrices} \\ \text{b) } [E+K_2'(z)][E+K_1''(z)]^{-1} \text{ in } D^- \text{ in case (9b);} \\ \text{c) } [E+K_2''(z)][E+K_1'(z)]^{-1} \text{ in } D^+ \text{ in case (9c).} \end{array}\right\} \qquad (10)$$

If, however, conditions (10) are not satisfied, then the solution of equation (8), (9) can be constructed by means of a finite number of linear transformations.

§ 2. Suppose that the kernels of the systems are such ($k(x) = k_+(x) - k_-(x)$):

$$\begin{aligned} &k^{\alpha}_{ij+}(x)e^{-yx} \in L(-\infty,\infty), \qquad y \geqslant a^{\alpha}_{ij}, \\ &k^{\alpha}_{ij-}(x)e^{-yx} \in L(-\infty,\infty), \qquad y \leqslant b^{\alpha}_{ij}. \end{aligned} \qquad (11)$$

System (A). We seek the solutions $f_j(x)$ of the system in the classes:

$$\begin{aligned} &f_{j+}(x)e^{-yx} \in L^2(-\infty,\infty), \qquad y \geqslant \min(b^1_{1j},\dots,b^1_{nj}) \equiv \alpha_j; \\ &f_{j-}(x)e^{-yx} \in L^2(-\infty,\infty), \qquad y \leqslant \max(a^2_{1j},\dots,a^2_{nj}) \equiv \beta_j. \end{aligned}$$

The free terms may belong to the following maximally broad admissible classes:

$$g_{i+}(x)e^{-yx} \in L^2(-\infty,\infty),$$

$$y \geqslant \max(a^1_{i1},\dots,a^1_{in},a^2_{i1},\dots,a^2_{in},\alpha_1,\dots,\alpha_n) \equiv a_i;$$

$$g_{i-}(x)e^{-yx} \in L^2(-\infty,\infty),$$

$$y \leqslant \min(b^1_{i1},\dots,b^1_{in},b^2_{i1},\dots,b^2_{in},\beta_1,\dots,\beta_n) \equiv b_i.$$

The Fourier transform of system (A) leads to a boundary-value problem of a special kind, whose boundary condition is given, in the general case, on $r \leqslant 2k+2$ ($k \leqslant n$) straight lines, where $2k$ is equal to the number of distinct constants $\alpha_j, \beta_j;\ j = 1,2,\dots,n$.

Special cases.

a) If all elements $k^{\alpha}_{ij}(x)$ of the i-th row of the matrix $k_{\alpha}(x)$ belong to one and the same class, i.e. $a^{\alpha}_{ij} = a^{\alpha}_i$, $b^{\alpha}_{ij} = b^{\alpha}_i$, $i,j = 1,2,\dots,n$, then the classes of solutions $f_j(x)$ do not depend on the index j, i.e. $\alpha_j \equiv \alpha$, $\beta_j \equiv \beta$, $j = 1,2,\dots,n$. In this case $2k \leqslant 2$ and $r \leqslant 4$.

b) If all elements $k_{ij}^{\alpha}(x)$ of the j-th column of the matrix $k_\alpha(x)$ belong to one and the same class, i.e. $a_{ij}^{\alpha} = a_j^{\alpha}$, $b_{ij}^{\alpha} = b_j^{\alpha}$, then the classes of the free terms—the functions $g_i(x)$—do not depend on the index i of the equation, i.e. $a_i = a$, $b_i = b$, $i = 1, \dots, n$. In this case $k \leqslant n$ and $r \leqslant 2n + 2$.

The solution of the resulting problem is unknown in general form. In the cases where analytic continuability of the matrices $K_\alpha^+(z)$, $K_\alpha^-(z)$ is assumed,

and of the vectors $G^+(z), G^-(z)$ inside certain vector-strips,* where they are meromorphic in character, this problem can be reduced to a "modified" Riemann boundary-value problem, which differs from the ordinary one $(')$ in that its solution—the vector $\Phi(z)$—includes a prescribed matrix multiplier of the form

$$T(z) \equiv T_p(z) = \prod_{k=1}^{m_p} \left[T_k^{(p)}(z)\right]^{\frac{1+\delta_p}{2}} \left\| \left(\frac{z - z_k^{(p)}}{z - iz_p} \right)^{\nu_{kj}^{(p)}} \right\|^{\delta_p},$$

where $\|R_j(z)\|$ is a diagonal matrix; $\delta_p = -1, 1$; $p = 0, 1, 2$; $\nu_{kj}^{(p)} \geq 0$; $z, z_k^{(p)} \in D_p$; D_1, D_2, D_0 are, respectively, the domains $\operatorname{Im} z > \beta$, $\operatorname{Im} z < \alpha$, $\alpha < \operatorname{Im} z < \beta$; $z_1 = z_2 = i\mu \in D_0$, $z_0 = -i\lambda \in D_2$, and $-\lambda < \min \operatorname{Im} z_k^{(2)}$ $(k = 1, 2, \dots, m_2)$; $T_k^{(p)}(z)$ is a matrix analytic in D_p with constant determinant different from zero, having zero order at infinity. The vector $\Phi(z)$ is holomorphic everywhere in the domain $D_0 + D_1 + D_2$, except for the points $z_k^{(p)}$ (when $\delta_p = -1$), where it is meromorphic in character.

By introducing a new unknown vector $\widetilde{\Phi}(z)$: $\Phi(z) = T(z)\widetilde{\Phi}(z)$, the "modified" problem reduces to the ordinary Riemann problem. In the "modified" problem the zeros of the components of the vector $\Phi(z)$ are decreased, and the poles increase the total index of the problem.

System (B). We seek solutions $f_j(x)$ of the system in the classes

$$f_{j+}(x)e^{-yx} \in L^2(-\infty, \infty),$$

$$y \geq \min\left(b_{1j}^1, \dots, b_{nj}^1,\ b_{1j}^2, \dots, b_{nj}^2\right) \equiv \alpha_j;$$

$$f_{j-}(x)e^{-yx} \in L^2(-\infty, \infty),$$

$$y \leq \max\left(a_{1j}^1, \dots, a_{nj}^1,\ a_{1j}^2, \dots, a_{nj}^2\right) \equiv \beta_j.$$

Admissible classes of free terms:

$$g_{i+}(x)e^{-yx} \in L^2(-\infty, \infty),$$

$$y \geq \max\left(a^1_{i1}, \dots, a^1_{in},\ \alpha_1, \dots, \alpha_n\right) \equiv a_i;$$

$$g_{i-}(x)e^{-yx} \in L^2(-\infty, \infty),$$

$$y \leq \min\left(b^2_{i1}, \dots, a^2_{in},\ \beta_1, \dots, \beta_n\right) \equiv b_i.$$

By modifying somewhat the method of investigation, we obtain for system (B) results analogous to those obtained above for system (A).

I consider it my pleasant duty to express my deep gratitude to F. D. Gakhov for supervising the present work.

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* If the elements of the i-th row of the matrix $K(z)$ are prescribed in the domain D_i, we shall say that the matrix $K(z)$ is prescribed in the vector-domain

$$D = \{D_1, D_2, \dots, D_n\}.$$

Note: Figure translations are in progress. See original paper for figures.