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Abstract

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ON THE MUTUAL POSITION OF THE ZEROS OF SUCCESSIVE POLYNOMIALS LEAST DEVIATING FROM ZERO

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Among the many remarkable properties possessed by the Chebyshev polynomials
$T_n(x) = \cos n \arccos x$, the following is known: the zeros of two successive polynomials $T_n(x)$ and $T_{n+1}(x)$ are mutually separated; the points of maximum deviation of both polynomials are likewise mutually separated. This assertion is easily verified directly, since for the polynomial $T_n(x)$ we know explicit expressions for its zeros $\left(x_{k,n} = \cos \frac{2k-1}{n}\pi,\ 1 \le k \le n\right)$ and for the points of its maximum deviation $\left(\xi_{k,n} = \cos \frac{k\pi}{n},\ 0 \le k \le n\right)$.

In the present note a generalization of the fact under consideration is given under sufficiently broad assumptions concerning the functions introduced below.

Let $f(x)$ be a function continuous together with its first derivative on the interval $[a,b]$; a zero x_0 of the function $f(x)$ will be called simple if $f'(x_0) \neq 0$; otherwise the zero x_0 will be called double.

Theorem. Let on the interval $[a,b]$ there be given two functions $t_n(x)$ and $t_{n+1}(x)$, continuous together with their first derivatives, possessing the following properties:

1) $t_n(x)$ has n simple zeros in the interval (a,b):

$$x_1 < x_2 < \cdots < x_n;$$

$t_{n+1}(x)$ has $n+1$ simple zeros in the interval (a,b):

$$y_1 < y_2 < \cdots < y_{n+1};$$

2) every linear combination $\lambda t_n(x) + \mu t_{n+1}(x)$ (λ, μ real, $\lambda^2 + \mu^2 \neq 0$) has $\le n+1$ zeros on the interval $[a,b]$;

3) on the interval $[a,b]$

$$|t_n(x)| \le 1, \qquad |t_{n+1}(x)| \le 1,$$

and, moreover, at $n+1$ distinct points $\xi_1 < \xi_2 < \dots < \xi_{n+1}$ and at $n+2$ distinct points $\eta_1 < \eta_2 < \dots < \eta_{n+2}$ the relations

$$t_n(\xi_k) = (-1)^k \qquad (k = 1, \dots, n+1), \tag{1}$$

$$t_n(\eta_k) = (-1)^k \qquad (k = 1, \dots, n+2). \tag{2}$$

Then the inequalities

$$a \le \eta_1 \le \xi_1 < \eta_2 < \xi_2 < \dots < \eta_{n+1} < \xi_{n+1} \le \eta_{n+2} \le b, \tag{3}$$

$$a < y_1 < x_1 < y_2 < \dots < x_n < y_{n+1} < b \tag{4}$$

are valid.

Inequalities (3) are closely connected with S. N. Bernstein' s theorem ((1), p. 87) on the points of maximum deviation of the polynomials least deviating from zero, formed from Chebyshev systems. These inequalities (3) can be established by means of arguments close to those given in the monograph (1). We shall, however, give here a somewhat different proof*, with a view to using it in deriving inequalities (4).

First we shall show that inequalities (3) hold. To this end consider the function

$$\Phi(x) = t_{n+1}^2(x) - t_n^2(x). \tag{5}$$

By assumption 2), the function $\Phi(x)$ can vanish on $[a, b]$ at most $2n+2$ times. From equalities (2) it follows that in each interval (η_k, η_{k+1}) there lies one zero of the function $t_{n+1}(x)$, namely y_k. Since from (5) it follows that, for all k,

$$\Phi(\eta_k) \ge 0, \qquad \Phi(\xi_k) \le 0, \tag{6}$$

then on the segment $[\eta_k, \eta_{k+1}]$ we have

$$\Phi(\eta_k) \ge 0, \qquad \Phi(y_k) \le 0, \qquad \Phi(\eta_{k+1}) \ge 0 \tag{7}$$

and, consequently, on this segment the function $\Phi(x)$ has at least two zeros.

Taking further into account that if an interior point η_k of the interval (a, b) is a zero of the function $\Phi(x)$, then η_k will be a double zero of this function, we conclude that the function $\Phi(x)$ has $2n+2$ zeros on the segment $[\eta_1, \eta_{n+2}]$. It is proved analogously that on each segment $[\xi_k, \xi_{k+1}]$ there lie ≥ 2 zeros and that on the segment $[\xi_1, \xi_{n+1}]$ there lie not fewer than $2n$ zeros of $\Phi(x)$.

We now show that

$$\eta_1 \le \xi_1, \qquad \xi_{n+1} \le \eta_{n+2}. \tag{8}$$

This is obvious if $\eta_1 = a,\ \eta_{n+2} = b$. Let $\eta_1 > a$; suppose, contrary to (8), that $a \leq \xi_1 < \eta_1$. Then from (6) it would follow that in the interval $[\xi_1, \eta_1)$ there must necessarily lie a zero of $\Phi(x)$, but this is impossible, since all $2n+2$ zeros of this function lie on the segment $[\eta_1, \eta_{n+2}]$. The second of the inequalities is proved analogously.

Let us verify that any segment $[\eta_k, \eta_{k+1}]$ $(k = 2, \ldots, n)$ can contain ≤ 1 points ξ_j. If $\eta_k = \xi_j,\ \eta_{k+1} = \xi_{j+1}$, then these points would be double zeros of $\Phi(x)$, and since, moreover, $\Phi(x_i) \geq 0,\ \Phi(y_i) \leq 0$, there would lie ≥ 5 zeros on the segment $[\eta_k, \eta_{k+1}]$, whence it follows that on the segment $[\eta_1, \eta_{n+2}]$ there would have to be $\geq 2n+3$ zeros of $\Phi(x)$.

If $\eta_k = \xi_j < \xi_{j+1} < \eta_{k+1}$, then the interval $[\eta_k, \eta_{k+1})$ contains ≥ 4 zeros, whence it again follows that the total number of zeros is $\geq 2n+3$. In exactly the same way, from the assumption $\eta_k < \xi_j < \xi_{j+1} < \eta_{k+1}$ it follows that either inside the interval (η_k, η_{k+1}) there are ≥ 4 zeros of $\Phi(x)$, or on the segment $[\eta_k, \eta_{k+1}]$ there are ≥ 5 zeros.

Further we have

$$\xi_1 < \eta_2, \qquad \eta_{n+1} < \xi_{n+1}. \tag{9}$$

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* A similar device was used by me in the note (2).

Indeed, if $\eta_2 \leqslant \xi_1$, then, since the segment $[\eta_2, \eta_{n+1}]$ can contain $\leqslant n-1$ points ξ_j, we would have $\eta_{n+1} < \xi_n < \xi_{n+1} \leqslant \eta_{n+2}$, and in the interval $(\eta_{n+1}, \eta_{n+2}]$ there would lie $\geqslant 3$ zeros of $\Phi(x)$, which leads to a contradiction. In exactly the same way we obtain that $\eta_{n+1} < \xi_{n+1}$.

Let us also establish the inequalities

$$\eta_2 < \xi_2, \qquad \xi_n < \eta_{n+1}. \tag{10}$$

Indeed, if $\eta_2 > \xi_2$, then in the interval $[\eta_1, \eta_2)$ there would lie $\geqslant 3$ zeros of $\Phi(x)$, which is impossible. From the hypothesis that $\eta_2 = \xi_2$, it follows that $\eta_3 < \xi_3 < \eta_4$, whence we conclude that either in the interval $[\eta_1, \eta_4]$ there lie $\geqslant 7$ zeros, or on the segment $[\eta_1, \eta_4]$ there lie $\geqslant 8$ zeros of $\Phi(x)$, but each of these conclusions leads to a contradiction. Analogously it is proved that $\xi_n < \eta_{n+1}$.

Thus we have shown, among other things, that each of the segments $[\eta_k, \eta_{k+1}]$ $(k = 1, \ldots, n+1)$ contains one and only one point ξ_j.

From (8), (9), and (10) we have $\eta_1 \leqslant \xi_1 < \eta_2 < \xi_2$. Suppose by induction that the inequalities

$$\eta_1 \leqslant \xi_1 < \eta_2 < \xi_2 < \cdots < \eta_r < \xi_r$$

have already been proved for some r ($2 \leqslant r \leqslant n$). We shall show that then

$$\xi_r < \eta_{r+1} < \xi_{r+1}.$$

If $\xi_r \geqslant \eta_{r+1}$, then on the segment $[\eta_{r+1}, \eta_{n+2}]$ there would lie $n+2-r$ points ξ_j; consequently, on at least one of the $n+1-r$ segments $[\eta_{r+k}, \eta_{r+k+1}]$ ($k = 1, \dots, n+1-r$) there would fall 2 points ξ_j, which cannot occur. In exactly the same way, if $\xi_{r+1} \leqslant \eta_{r+1}$, then on the segment $[\eta_1, \eta_{r+1}]$ there would fall $r+1$ points ξ_j, and, consequently, on one of the r segments $[\eta_k, \eta_{k+1}]$ ($k = 1, \dots, r$) there would fall 2 points ξ_j. Thus, the inequalities (3) are proved.

We pass to the proof of the inequalities (4). We first show that $y_1 < x_1$, $x_n < y_{n+1}$.

Assume, to the contrary, that $x_1 \leqslant y_1$. Then, by virtue of the inequalities

$$\Phi(\eta_1) \geqslant 0, \qquad \Phi(\xi_1) \leqslant 0, \qquad \Phi(x_1) \geqslant 0, \qquad \Phi(y_1) \leqslant 0, \qquad \Phi(\eta_2) \geqslant 0,$$

either in the interval $[\eta_1, \eta_2)$ there will lie $\geqslant 3$ zeros of $\Phi(x)$, or on the segment $[\eta_1, \eta_2]$ there will be $\geqslant 4$ zeros; this, as was shown in the derivation of the inequalities (3), leads to a contradiction. Further, from (3) and the obvious inequalities $\xi_k < x_k < \xi_{k+1}$, $\eta_k < y_k < \eta_{k+1}$, we directly obtain

$$x_{k-2} < \xi_{k-1} < \eta_k < y_k < \eta_{k+1} < \xi_{k+1} < x_{k+1} \qquad (k = 3, \dots, n-1),$$

i.e. $x_{k-2} < y_k < x_{k+1}$. Let us establish that

$$x_{k-1} < y_k < x_k.$$

It follows from (5) that $\Phi(y_k) \leqslant 0$. If $\Phi(y_k) = 0$, then y_k is a double zero of the function $\Phi(x)$, and hence y_k coincides with one of the two points x_{k-1} or x_k. Suppose, for definiteness, that $y_k = x_k$; then the inequalities

$$\eta_k < y_k = x_k < \xi_{k+1} < \eta_{k+1},$$

$$\Phi(\eta_k) \geqslant 0, \qquad \Phi(y_k) = 0, \qquad \Phi(\xi_{k+1}) \leqslant 0, \qquad \Phi(\eta_{k+1}) \geqslant 0,$$

from which it follows that on the interval $[x_k, \eta_{k+1}] \subset (\eta_k, \eta_{k+1}]$ there must be ≥ 3 zeros if $\Phi(\eta_k) > 0$, or ≥ 4 zeros if $\Phi(\eta_k) = 0$. Similarly it is proved that $y_k \neq x_{k-1}$. Thus, $\Phi(y_k) < 0$. If $x_k < y_k$, then from the inequalities

$$\eta_k < \xi_k < x_k < y_k < \eta_{k+1},$$

$$\Phi(\eta_k) \geq 0, \qquad \Phi(\xi_k) \leq 0, \qquad \Phi(x_k) > 0, \qquad \Phi(y_k) < 0, \qquad \Phi(\eta_{k+1}) \geq 0$$

it would follow that in the interval (η_k, η_{k+1}) there must lie ≥ 4 zeros. In the same way we obtain the inequality $x_{k-1} < y_k$. The theorem is completely proved*.

If one does not assume differentiability of the functions $t_n(x)$ and $t_{n+1}(x)$, and defines the multiplicity of zeros as in the monograph ([1]) (p. 8), then in inequalities (3) and (4) the signs $<$ must, generally speaking, be replaced by the signs $\leq$.

Atkinson ([3]) recently considered polynomials

$$p_n(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

least deviating from zero on the interval $[a, b]$, with a continuous and positive weight $w(x)$ on this interval. He established that the zeros of two consecutive polynomials $p_n(x)$ and $p_{n+1}(x)$ are mutually separated. This assertion is a special case of the theorem proved. Indeed, if in the preceding arguments one adopts the usual algebraic method of counting multiplicities of zeros of $t_n(x) = w(x)p_n(x)$, then we immediately arrive at the conclusion that, in the case where $w(x)$ is not differentiable, although in inequalities (3) in some places the signs $<$ may have to be replaced by the signs $\leq$, in inequalities (4) the signs of strict inequality are preserved everywhere.

In addition to the natural application of our theorem to two consecutive polynomials

$$t_n(x) = \sum_0^n a_k \varphi_k(x), \qquad t_{n+1}(x) = \sum_0^{n+1} a_k \varphi_k(x),$$

least deviating from zero, where $\{\varphi_\nu(x)\}_{\nu=0}^n$ and $\{\varphi_\nu(x)\}_{\nu=0}^{n+1}$ form Chebyshev systems (([1]), p. 87), we indicate here still another example. In the investigation of the question of the best approximation of $|x|$, S. N. Bernstein (([1]), p. 58) considered oscillator polynomials $P_k(x)$ and $P_{k+1}(x)$ from the Descartes system $\{\varphi_\nu(x)\}_{\nu=0}^n$, deprived, respectively, of the functions $\varphi_k(x)$ and $\varphi_{k+1}(x)$. From the theorem proved it follows that the zeros and the points of maximum deviation of $P_k(x)$ alternate, respectively, with the zeros and points of maximum deviation of $P_{k+1}(x)$.

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CITED LITERATURE

1. S. N. Bernstein, *Extremal Properties of Polynomials and the Best Approximation of Continuous Functions of One Real Variable*, Moscow–Leningrad, 1937.

2. V. S. Videnskii, DAN, 67, No. 5 (1949).

3. F. V. Atkinson, Proc. Am. Math. Soc., 7, No. 2, 267 (1956).

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* This theorem was proved at the end of 1946; at the same time it was presented at S. N. Bernstein' s seminar at Moscow State University and was included in my diploma thesis, defended in 1947.

Note: Figure translations are in progress. See original paper for figures.