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Abstract

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ON MAPPINGS OF THE PRODUCT OF A TOPOLOGICAL SPACE BY A EUCLIDEAN SPACE INTO A EUCLIDEAN SPACE

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In this note X is a topological space; R_y^q, R_u^q are q-dimensional Euclidean oriented spaces; $Z = [X, R_y^q]$ is the topological product of X by R_y^q; x, y, z, u (possibly with an index) are points respectively of the spaces X, R_y^q, Z, R_u^q; $O^x = O^x(x_0)$, $O^y = O^y(y_0)$, $O^z = O^z(z_0) = O^z(x_0, y_0)$, $O^u = O^u(u_0)$ are absolute neighborhoods of the points $x_0, y_0, z_0 = (x_0, y_0), u_0$ in the spaces X, R_y^q, Z, R_u^q, respectively; the same notations are adopted for absolute neighborhoods of subsets of the spaces X, R_y^q, Z, R_u^q relative to these spaces; π_x, π_y are the projections of the space Z respectively onto the spaces X, R_y^q; G is a nonempty open subset of the space Z; f is a continuous mapping of G into R_u^q; (x_0, y_0) is a point of G; $u_0 = f(x_0, y_0)$; F^u is a closed subset of the space R_u^q; Φ^z is a compact subset $\subset G$; Λ is the empty set; M is a subset of the space Z.

In addition, the following notation and definitions are used:

1) $M(x) = \{y : (x, y) \in M\}$.
2) $\widetilde{M} = \{(x, y) : (x, y) \in \overline{M},\ x \in \pi_{xM}\} = \{(x, y) : x \in \pi_{xM},\ y \in \overline{M(x)}\} = \overline{M} \cap \pi_x^{-1}\pi_{xM}$.
3) M is locally bounded with respect to y if, for every $x_1 \in \pi_{xM}$, there is an $O^x = O^x(x_1)$ such that $\pi_y(M \cap \pi_x^{-1}O^x)$ is bounded in R_y^q.
4) M is connected with respect to y if $M(x)$ is connected for every $x \in \pi_{xM}$.
5) f_x is the mapping of $G(x)$ into R_u^q given by the formula $f_x(y) = f(x, y)$ $(y \in G(x))$.
6) $\hat{f}$ is the mapping of G into $[X, R_u^q]$ given by the formula $\hat{f}(x, y) = (x, f(x, y))$.
7) $E_g = (E)_g$ is the boundary of the set E relative to that one of the spaces X, R_y^q, Z, R_u^q which contains it.
8) $m(E)$ is equal to the cardinality of the set E if E is finite or Λ, and is equal to $+\infty$ if E is infinite.

9) $k(\Phi^z(x), f_x) = \sup m(\Phi^z(x) \cap f_x^{-1}u)$.

10) $k(\Phi^z, f) = \sup_{x \in X}(\Phi^z(x), f_x)$.

11) If f_{x_0} is differentiable at the point y_0, then $J(f_{x_0}, y_0)$ denotes the Jacobian of the mapping f_{x_0} at the point y_0.

12) If, for every $x \in \pi_{xG}$, the mapping f_x is differentiable on $G(x)$, then we put

$$G^+ = \{(x,y) : (x,y) \in G,\ J(f_x, y) > 0\}, \quad G^- = \{(x,y) : (x,y) \in G,\ J(f_x, y) < 0\}, \quad G^0 = \{(x,y), (x,y)$$

13) y_0 is an isolated point of the mapping f_{x_0} if, for all sufficiently···

for sufficiently small $h \neq 0$ we have $f_{x_0}(y_0 + h) \neq f_{x_0}(y_0)$. In this case the local degree $\gamma(f_{x_0}, y_0)$ is defined, equal to the degree $\gamma(O^y, f_{x_0}, u_0)$, where $O^y = O^y(y_0)$ is sufficiently small. If f_{x_0} is differentiable at y_0, and $J(f_{x_0}, y_0) \neq 0$, then, as is known, y_0 is an isolated point of the mapping f_{x_0}, and we have $\gamma(f_{x_0}, y_0) = \operatorname{sign} J(f_{x_0}, y_0)$.

§ 1. In this section G is locally, with respect to y; f is continuous in $\widetilde{G}$.

Theorem 1. *Let* $\overline{G}(x_0) \cap f_{x_0}^{-1}F^u \subseteq G(x_0)$. *Then there exists* $O^x = O^x(x_0) \subseteq \pi_{xG}$ *such that for all* $x \in O^x$ *we have:*

1.1) $\overline{G}(x) \cap f_x^{-1}F^u \subseteq G(x)$;

1.2) *for every* $u \in F^u$ *the degree* $\gamma(G(x), f_x, u) = \gamma(G(x_0), f_{x_0}, u)$ *is defined.*

Corollary 1. *Let* $\overline{G}(x_0) \cap f_{x_0}^{-1}F^u \subseteq G(x_0)$, *and suppose* $\gamma(G(x_0), f_{x_0}, u) \neq 0$ *for all* $u \in F^u$. *Then there exists* $O^x = O^x(x_0)$ *such that* $F^u \subseteq f_{xG}(x)$ *for all* $x \in O^x$.

Corollary 2. *Let* C^x *be connected,* $C^x \subseteq \pi_{xG}$, *and suppose* $\overline{G}(x) \cap f_x^{-1}u_0 \subseteq G(x)$ *for all* $x \in C^x$. *Then* $\gamma(G(x), f_x, u_0) = \text{const}$ *for all* $x \in C^x$.

Corollary 3. *If* $F^u \cap f_{x_0}\overline{G}(x_0) = \Lambda$, *then there exists* $O^x = O^x(x_0) \subseteq \pi_{xG}$ *such that for all* $x \in O^x$, $u \in F^u$, *the degree* $\gamma(G(x), f_x, u) = 0$ *is defined.*

Theorem 2. *Let* $\overline{G}(x_0) \cap f_{x_0}^{-1}u_0 \subseteq G(x_0)$, $\gamma(G(x_0), f_{x_0}, u_0) \neq 0$. *Then for every sufficiently small* $O^u = O^u(u_0)$ *there is an* $O^z = O^z([x_0, \overline{G}(x_0) \cap f_{x_0}^{-1}u_0]) \subseteq G$ *such that for all* $x \in \pi^z_{xO}$ *we have:*

2.1) $f^z_{xO}(x) = O^u$;

2.2) $\overline{G}(x) \cap f_x^{-1}O^u = O^z$;

2.3) *for every* $u \in O^u$ *the degree* $\gamma(O^z(x), f_x, u) = \gamma(G(x_0), f_{x_0}, u_0)$ *is defined.*

Remark 1. All the results of the present section are, as far as I know, essentially new even in the case $X = R^p$ with $p > 0$. For the case $p = 0$ (f is a mapping of G, open in R^q_y, into R^q_u), analogous results were obtained by me in (^{1},^{2}). I note that no restrictions are imposed on the space X, except the fulfillment of the four Kuratowski axioms; it may even fail to be a T_0-space and need not satisfy the first axiom of countability. Also of interest is the case when X is an arbitrary subset of the space R^p, and G is a set open in $[X, R^q_y]$.

§ 2. In this section y_0 is an isolated point of the mapping f_{x_0}, and $\gamma(f_{x_0}, y_0) \neq 0$.

Theorem 3. *Let $O^z = O^z(x_0, y_0)$ be given. Then for every sufficiently small (connected) $O^u = O^u(u_0)$ there is a (connected with respect to y) $O^z_1 = O^z_1(x_0, y_0)$, contained in O^z and such that:*

3.1) $\widetilde{O}^z_1 \subseteq G$;

3.2) $\overline{O}^z_1$ is locally bounded with respect to y;

3.3) $\overline{O}^z_1(x_0) \cap f^{-1}_{x_0} u_0 = \{y_0\}$;

3.4) $f_{xO}1^z(x) = O^u$ for every $x \in \pi_{xO}1^z$;

3.5) $\overline{O}^z_1(x) \cap f^{-1}_x O^u = O^z_1(x) (\overline{O}^z_1(x) \cap f^{-1}_x O^u = O^z_1(x))$ for every $x \in \pi_{xO}1^z$;

3.6) for any $x \in \pi_{xO}1^z$, $u \in O^u$ the degree $\gamma(O^z_1(x), f_x, u) = \gamma(f_{x_0}, y_0)$ is defined;

3.7) $\hat{f}\,\widetilde{O}^z_1 = [\pi_{xO}1^z, O^u]$;

3.8) $\hat{f}\,(\widetilde{O}^z_1 \setminus O^z_1) = [\pi_{xO}1^z, \overline{O}^u \setminus O^u]$ $(\hat{f}\,\widetilde{O}^z_1 = [\pi_{xO}1^z, \overline{O}^u])$;

3.9) if $x \in \pi_{xO}1^z$, and f_x is differentiable on $O^z_1(x)$, then

$$\operatorname{mes}\{y : y \in O^z_1(x),\ \operatorname{sign} J(f_x, y) = \operatorname{sign} \gamma(f_{x_0}, y_0)\} > 0,$$

$$\operatorname{mes} f_x\{y : y \in O^z_1(x),\ \operatorname{sign} J(f_x, y) = \operatorname{sign} \gamma(f_{x_0}, y_0)\} = \operatorname{mes} O^u.$$

Corollary 1. The mapping $\hat{f}$ is open at the point (x_0, y_0).

Remark. All the results of the present paragraph are, as far as I know, essentially new even in the case $X = R^p$ for $p > 0$. For the case $p = 0$ (f is a mapping of an open subset G in R^q_y into R^q_u) analogous results were obtained by me in (^{3,4}) under the additional assumption that f is differentiable in G.

§ 3. In this paragraph the mapping f_x is differentiable in $G(x)$ for every $x \in \pi_x G$.

Theorem 4. Each of the sets $\pi_x G^+$, $\pi_x G^-$ is open in the space X.

Theorem 5. If $G^0 = \Lambda$, then each of the sets G^+, G^- is open in the space Z.

Corollary 1. Let $G^0 = \Lambda$, and let C be a connected subset of the set G. Then $\operatorname{sign} J(f_x, y) = \text{const}$ for all $(x, y) \in C$.

Corollary 2. If $G^0 = \Lambda$, and G is a domain, then $\operatorname{sign} J(f_x, y) = \text{const}$ for all $(x, y) \in G$.

Theorem 6. Let $O^z = O^z(x_0, y_0) \subseteq G \setminus G^0$. Then for every sufficiently small connected $O^u = O^u(u_0)$ there exists a locally bounded in y and connected in y neighborhood $O_1^z = O_1^z(x_0, y_0)$, contained in O^z, such that the mapping φ, defined on the set $[\pi_x O_1^z, O^u]$ by the formula

$$\varphi(x,u) = O_1^z(x) \cap f_x^{-1} u \qquad ((x,u) \in [\pi_x O_1^z, O^u]),$$

is a single-valued continuous open mapping of the set $[\pi_x O_1^z, O^u]$ into the space R_y^q, and we have:

6,1) for every $x \in \pi_x O_1^z$ the mapping f_x is a differentiable topological mapping of $O_1^z(x)$ onto O^u;

6,2) for every fixed $x \in \pi_x O_1^z$ the mapping φ_x ($\varphi_x(u) = \varphi(x,u)$) is a differentiable topological mapping of O^u onto $O_1^z(x)$, inverse to the mapping f_x, considered on $O_1^z(x)$;

6,3) for every fixed $u \in O^u$ the mapping φ_u ($\varphi_u(x) = \varphi(x,u)$) is a continuous mapping of $\pi_x O_1^z$ into R_y^q, and we have:

$$f(x, \varphi_u(x)) = u \qquad (x \in \pi_x O_1^z),$$

i.e. φ_u is the unique continuous implicit mapping determined from the equation

$$f(x,y) = u \qquad ((x,y) \in O_1^z);$$

6,4) $\hat{f}$ is a topological mapping of O_1^z onto $[\pi_x O_1^z, O^u]$;

6,5) $\varphi(x,u) = \pi_y(\hat{f}^{-1}(x,u))$ for all $(x,u) \in [\pi_x O_1^z, O^u]$.

Theorem 7. Let $q = 1$, and let $O^z = O^z(x_0, y_0)$ be such that for every $x \in \pi_x O_1^z$ the set $(O^z \cap G^0)(x)$ contains not more than one point. Suppose, further, that k is a nonnegative integer such that the function f_x is differentiable $2k$ times in some neighborhood of the point y_0 and has a finite derivative of order $2k+1$ at the point y_0, and moreover

$$\frac{d^l f_{x_0}}{dy^l}(y_0) = 0 \qquad (1 \le l \le 2k), \qquad \frac{d^{2k+1} f_{x_0}}{dy^{2k+1}}(y_0) \neq 0.$$

Then for every sufficiently small interval $O^u = O^u(u_0)$ there exists a locally bounded in y and connected in y neighborhood $O_1^z = O_1^z(x_0, y_0)$,

contained in O^z and such that the function φ, defined on the set $[\pi_x O_1^z, O^u]$ by the formula

$$\varphi(x,u) = O_1^z(x) \cap f_x^{-1} u \qquad ((x,u) \in [\pi_x O_1^z, O^u]),$$

effects a one-to-one continuous open mapping of the set $[\pi_x O_1^z, O^u]$ onto the line R_y^1; moreover assertions 6, 1)–6, 5) hold.

Theorem 8. Let $G^0 = \Lambda$; let X be a Hausdorff space satisfying the first axiom of countability. Then $k(\Phi^z, f) < +\infty$.

Theorem 9. Let $G^0 = \Lambda$; let X be a metric space with a countable base, and suppose that an outer Carathéodory measure μ_x is given in X such that every point $x \in X$ has a neighborhood of finite μ_x-measure. By μ_y, μ_u denote the q-dimensional Lebesgue measures, respectively, in the spaces R_y^q, R_u^q. Further, let $\varphi(u)$ be a complex function, defined on $f\Phi^z$ (almost everywhere on $f\Phi^z$) and summable on $f\Phi^z$. Finally, suppose that

$$\inf_{(x,y)\in\Phi^z} \operatorname{vrai} |J(f_x, y)| = m > 0. \tag{1}$$

Then the function $\varphi(f(x, y))$ is defined on Φ^z (almost everywhere on Φ^z), is summable on Φ^z, and

$$\int_{\Phi^z} |\varphi(f(x, y))| \, d(\mu_x \times \mu_y) \le \frac{2}{m} k(\Phi^z, f) \, \mu_x(\pi_x, \Phi^z) \int_{f\Phi^z} |\varphi(u)| \, d\mu_u < +\infty.$$

If, however, $\operatorname{sign} J(f_x, y) = \text{const}$ for all $(x, y) \in \Phi^z$, then the constant 2 on the right-hand side of the last inequality is replaced by the constant 1.

Remark. All the results of the present paragraph are essentially new even in the case $X = R^p$ with $p > 0$. For $p = 0$ (f is a mapping, open in R_y^q, of the set G into R_u^q) results analogous to Corollary 2 of Theorem 5 and Theorem 6 were obtained by me in (^{3}). Particular cases of Corollary 2 of Theorem 6 were obtained by L. D. Kudryavtsev in (^{5,6}).

Theorems 4, 5, and 7 are essentially new also in the case when $X = R^p$, and f is differentiable jointly in all the variables $x_1, \dots, x_p; y_1, \dots, y_q$.

Theorems 8 and 9 are essentially new in the case $X = R^p$, μ_x is p-dimensional Lebesgue measure, and under any assumptions concerning the smoothness of the mapping f. Moreover, if the partial derivatives of the mapping functions $\partial f_i / \partial y_j$ $(i, j = 1, \dots, q)$ are continuous jointly in the variables $x_1, \dots, x_p; y_1, \dots, y$, then the conditions $G^0 = \Lambda$ and (1) in Theorems 8 and 9 are unnecessary. It is enough to require that $\Phi^z \subseteq G \setminus G^0$.

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