ON PERMUTATIONS OF THE TRIGONOMETRIC SYSTEM

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Abstract

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ON PERMUTATIONS OF THE TRIGONOMETRIC SYSTEM

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Let $f(x) \in L(0, 2\pi)$ be a 2π-periodic function whose Fourier series is

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx), \tag{1}$$

where

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos kt \, dt \qquad (k = 0, 1, 2, \ldots);$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin kt \, dt \qquad (k = 1, 2, \ldots).$$

Definition. We shall call the series (1) unconditionally convergent almost everywhere on $[0, 2\pi]$ if it converges almost everywhere on $[0, 2\pi]$ after any permutation of its terms. In this case the set of points of divergence, generally speaking, depends on the order of the terms.

If we slightly modify our reasoning from the note ([6]), it is easy to see that the following theorem is true.

Theorem 1. Suppose

$$\sum_{n=10}^{\infty} \frac{(\ln \ln n)^{1+\varepsilon} \ln n}{n} \{E_n^{(2)}(f)\}^2 < \infty; \tag{2}$$

then the series (1) converges unconditionally almost everywhere on $[0, 2\pi]$, where $\varepsilon > 0$; $E_n^{(2)}(f)$ is the best approximation, in the L^2 metric, of the function $f(x)$ by trigonometric polynomials of order $n - 1$.

Hence, as a consequence, it follows that if the modulus of continuity of the function $f(x)$ satisfies the inequality

$$\omega(\delta, f) \leq \frac{1}{\ln \frac{1}{\delta} \left(\ln \ln \frac{1}{\delta}\right)^{1+\varepsilon}}, \qquad \varepsilon > 0,$$

then the series (1) converges unconditionally almost everywhere on $[0, 2\pi]$.

It is known that if

$$\sum_{n=10}^{\infty} \frac{1}{\sqrt{n}} E_n^{(2)}(f) < \infty, \tag{3}$$

then the series (1) converges absolutely on the interval $[0, 2\pi]$ (see, in this connection, the work of S. B. Stechkin ([4])). It is easy to see that condition (2) is considerably weaker than condition (3).

One can give a sufficient criterion for the unconditional convergence almost everywhere of the series (1) in a somewhat different form. Namely, using the result

Orlicz (([1]), p. 170) and one of the theorems of our paper ([5]), we are convinced of the validity of the following assertion:

Theorem 2. If for some $\varepsilon > 0$ the integral

$$\int_0^{2\pi} \int_0^{2\pi} \frac{|\ln t| \, |\ln|\ln t||^{1+\varepsilon}}{t} [f(x+t) - f(x-t)]^2 \, dx \, dt < \infty, \tag{4}$$

then the series (1) converges unconditionally almost everywhere on $[0, 2\pi]$.

Let us now consider the case when conditions (2) and (4) are certainly not satisfied.

In the paper of A. N. Kolmogorov and D. E. Menshov ([2]) there is the following assertion (belonging to A. N. Kolmogorov):

There exists a function $f(x) \in L^2(0, 2\pi)$ such that the terms of its Fourier series (1) can be rearranged in such a way that the resulting series diverges almost everywhere on $[0, 2\pi]$.

Somewhat later Marcinkiewicz ([3]) proved:

There exists a function $f(x) \in L^p(0, 2\pi)$ such that the terms of its Fourier series (1) can be rearranged in such a way that the resulting series has not a single subsequence of partial sums converging on a set of positive measure. Here $1 \leq p < 6/5$.

The last result can be generalized. We first give the following definition.

Definition. Let $T = \|C_{m,n}\|$ be a matrix defining a linear regular summability method T. We call the T-summability method a K-method if for every $m = 0, 1, 2, \ldots$

$$\sum_{n=0}^{\infty} n|C_{m,n}| = P(m) < \infty.$$

These summability methods were introduced, apparently, by Hardy and Rogozinski.

Now we can formulate our assertion.

Theorem 3. Let

$$\sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \tag{5}$$

be a trigonometric series such that

$$|a_k| \leq 1, \qquad |b_k| \leq 1 \qquad (k = 1, 2, \ldots)$$

and

$$\sum_{k=1}^{\infty}(a_{n_k}^2 + b_{n_k}^2) = \infty \qquad (n_1 < n_2 < \cdots),$$

where the sequence $\{n_k\}$ has the property: the equation $n = \pm n_i \pm n_j$ has no more than A solutions for all $n \neq 0$ (A is a constant). Suppose, moreover, that T is some K-summability method. Then the terms of the series (5) can be rearranged in such a way that the newly obtained series

$$\sum_{i=1}^{\infty}(a_{\nu_i} \cos \nu_i x + b_{\nu_i} \sin \nu_i x) \tag{6}$$

will not be summable by the method T on any set of positive measure, and also no subsequence of partial sums of the series (6) will converge on any set of positive measure.

Remark 1. By Banach' s theorem ([7], p. 215), among the trigonometric series (5) that satisfy the conditions of Theorem 3 there are Fourier series. Moreover, on the basis of a result of Marcinkiewicz, there are also Fourier series from the class L^p, where $1 \leq p < {}^6/_5$.

Starting from Theorem 3, it is easy to show that the following theorem is true.

Theorem 4. Let ε be any positive number. Then one can construct a function $f(x) \in L(0, 2\pi)$ such that $f(x) = 0$ for $x \in [\varepsilon, 2\pi - \varepsilon]$, and $f(x)$ has a continuous derivative of arbitrary order on any interval $[\delta, 2\pi - \delta]$ $(\delta > 0)$. Moreover, if T is some K-method of summation, then the terms of the Fourier series (1) of the function $f(x)$ can be rearranged in such a way that the newly obtained series (6) will not be summable by the method T on any set E of positive measure, and also no infinite subsequence of partial sums of the rearranged series will converge on any set of positive measure.

Remark 2. The function $f(x)$ can even be constructed from the class L^p, and in such a way that its Fourier series (1) converges everywhere on $[0, 2\pi]$, where $1 \le p < {}^6/_5$ (see Remark 1).

Corollary 1 (on Riemann' s localization principle). Suppose that $\{\cos n_i x, \sin n_i x\}$ is a rearranged trigonometric system and that $f(x) \in L(0, 2\pi)$, with $f(x) = 0$ for $x \in [a, b] \subset (0, 2\pi)$, and

$$\frac{a_0}{2} + \sum_{i=1}^{\infty} (a_{n_i} \cos n_i x + b_{n_i} \sin n_i x) \tag{7}$$

is the Fourier series of the function $f(x)$ with respect to the rearranged trigonometric system. Then the assertion that the series (7) will converge on (a, b) is, generally speaking, false. Moreover, the series (7) may even turn out not to be summable by Abel' s method on any set of positive measure. All the more, this holds for all Cesàro means.

This corollary follows directly from Theorem 4.

Corollary 2 (on unconditional convergence). If $f(x) \in L(0, 2\pi)$ and (1) is its Fourier series, then from the fact that $f(x)$ has arbitrarily good properties on the interval $[a, b] \subset (0, 2\pi)$, one cannot conclude that the series (1) converges unconditionally almost everywhere on $[a, b]$.

This corollary follows from Corollary 1.

Corollary 3 (on convergence in mean and in measure). If $f(x) \in L(0, 2\pi)$ and (7) is its Fourier series with respect to the rearranged trigonometric system, then, generally speaking, one cannot even assert that some subsequence of partial sums of the series (7) converges in mean L^q on some set E of positive measure for some $q > 0$. The same holds for convergence in measure (the function $f(x)$ may vanish on $[a, b] \subset (0, 2\pi)$).

Remark 3. In all the statements formulated by us, we may assume that $f(x) \in L^p(0, 2\pi)$, where $1 \le p < {}^6/_5$ (see Remark 2).

A certain supplement to Theorem 3 is Theorem 5.

Theorem 5. Let

$$\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{8}$$

be a trigonometric series such that

$$\sum_{n=1}^{\infty}(a_n^2 + b_n) = \infty.$$

Then the terms of the series (8) can be rearranged in such a way that the new series

$$\sum_{\nu=1}^{\infty}\left(a_{m_\nu} \cos m_\nu x + b_{m_\nu} \sin m_\nu x\right) \tag{9}$$

diverges without bound almost everywhere on $[0, 2\pi]$ and does not converge in the metric L on $[0, 2\pi]$.

In one of Orlicz' s papers (^{8a}) some results in this direction are contained for orthogonal systems.

From Theorem 5 there immediately follows a number of corollaries which may also be of independent interest.

Corollary 4. If the trigonometric series (8), after any rearrangement of its terms, has partial sums bounded (by a number A) on some set E of positive measure, then it is the Fourier series of some function $f(x) \in L^2(0, 2\pi)$ (the number A and the set E depend, generally speaking, on the rearrangement). In particular, if the series (8), after any rearrangement, converges on a set E of positive measure, then it is the Fourier series of a function $f(x) \in L^2(0, 2\pi)$. Therefore Corollary 5 is valid.

Corollary 5. If a function $f(x) \in L^p(0, 2\pi)$ $(p < 2)$ and $f(x) \notin L^2(0, 2\pi)$, then the Fourier series of the function $f(x)$ certainly cannot be unconditionally convergent almost everywhere, i.e., only Fourier series of class L^2 can be unconditionally convergent series.

This assertion can easily be obtained from Orlicz' s results (^{8b}).

Corollary 6. The rearranged trigonometric system $\cos m_\nu x$, $\sin m_\nu x$ is not, generally speaking, a basis in the space of functions $f(x) \in L^p(0, 2\pi)$, where $1 \le p < 2$.

This assertion can also be derived from Orlicz' s results (^{8v,g}).

Thus we obtain the following conclusion: Riemann' s localization principle, convergence of Fourier series almost everywhere, representation of a function by a trigonometric series in the metric L^p $(p < 2)$, etc., are closely connected with the "proper" arrangement of the trigonometric system, i.e., these properties are violated if we consider Fourier series with respect to the "rearranged" trigonometric system.

Remark 4. Results of an analogous type also hold for some other orthogonal systems.

Theorem 6. There exists a 2π-periodic continuous function $f(x)$ whose Fourier series, after some rearrangement, does not converge on $[0, 2\pi]$ in the metric L^q for any $q > 2$ (the rearrangement does not depend on q).

From Theorem 6 there follows Corollary 7.

Corollary 7. The rearranged trigonometric system, generally speaking, is not a basis in the metric L^p in the space of functions $f(x) \in L^p(0, 2\pi)$, where $p > 2$.

Remark 5. It is easy to show that one can find such a **fixed** rearranged trigonometric system which possesses at once all the properties indicated above.

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Note: Figure translations are in progress. See original paper for figures.