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Abstract

Full Text

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ON A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS

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1°. Let us agree on the following notation. We shall denote matrices by capital Latin letters, row vectors by lowercase letters, and scalar quantities by Greek letters. The exceptions will be: t—time, n—the order of vectors and matrices; V —a Lyapunov function; $dV/dt = -W$—its derivative. All matrices, vectors, and numbers are assumed to be real. The components of vectors will be arranged in the form of a column (notation a) or in the form of a row (notation a^*). Then $a^*b = (a, b)$ is the scalar product; ab^* is a matrix.

In the book of A. I. Lur' e ([1]), systems of differential equations of "indirect automatic control with one regulating element" are introduced and studied:

$$\frac{dx}{dt} = Ax + a \cdot \varphi(\sigma), \qquad \frac{d\sigma}{dt} = (b, x) - \rho\varphi(\sigma). \tag{1}$$

Here $\rho > 0$ is the "feedback coefficient" ; $\varphi(\sigma)$, the "servomotor characteristic," is assumed continuous and satisfying the conditions

$$\varphi(0) = 0, \qquad \sigma\varphi(\sigma) > 0 \quad \text{for } \sigma \neq 0. \tag{2}$$

In addition, we shall assume that the characteristic equation of the "open-loop system" $\det(A - \lambda E) = 0$ has all roots with negative real parts. Following A. I. Lur' e, we shall solve here the following problem: it is required to determine a solution $x = 0,\ \sigma = 0$ of system (1), for which the trivial solution $x = 0,\ \sigma = 0$ of system (1) is stable in the large, i.e. it is stable "in the small" in the sense of Lyapunov and, moreover, every solution $x \to 0,\ \sigma \to 0$ as $t \to \infty$.

The purpose of the present note is to present A. I. Lur' e' s method in invariant form. This makes it possible to write the system of resolving equations directly in terms of the coefficients of system (1)*. We also show here that a certain supplement to Lur' e' s method leads to sufficient conditions including all the sufficient conditions that can be obtained with the aid of the Lyapunov function indicated below of the form (3)**.

The algebraic calculations connected with obtaining concrete conditions on the coefficients in the general case for systems of order > 5 ($n \geq 4$) are extremely cumbersome; nevertheless, it can be shown that in principle they are always feasible.

2°. We shall seek, for system (1), a Lyapunov function of the form

$$V = (Hx, x) + \int_0^\sigma \varphi(\sigma)\, d\sigma, \qquad H^* = H. \tag{3}$$

* See, in this connection, the note in ([1]), p. 91.
** As far as the author knows, Lyapunov functions of another form have not been used for the class of problems under consideration. It may also be shown that the presence in the right-hand side of formula (3) of a term of the form $\sigma h^* x$ gives nothing new, since we would obtain $h = 0$.

Differentiation along the solutions of system (1) gives

$$\dot{V} = -W = -(x^* G x + 2g^* x + \gamma),$$

where $-G = A^* H + HA, \quad -g = (Ha + \frac{1}{2}b)\varphi, \quad \gamma = \rho\varphi^2.$

It can be proved that the usual condition $V > 0$, $W > 0$ for $|\sigma| + \|x\| \neq 0$ guarantees stability in the large.* In order that $W > 0$ for $|\sigma| + \|x\| \neq 0$, it is necessary and sufficient that** $\gamma G - gg^* = \frac{1}{4}\rho\varphi^2 G > 0$.

We arrive at the quadratic inequality for the matrix H

$$-\rho(A^* H + HA) - (Ha + \tfrac{1}{2}b)(Ha + \tfrac{1}{2}b)^* \equiv \tfrac{1}{2}\rho G > 0. \tag{4}$$

It is easy to show (([3]), pp. 430–431) that from inequality (4) it also follows that $H > 0$, i.e. $V > 0$ for $\|x\| + |\sigma| \neq 0$. Therefore, for stability in the large it is sufficient that equation (4), with some matrix $G > 0$, admit as a solution a symmetric matrix H.

Define the linear operator $Y = \mathfrak{A}(X)$ by the formula $A^* Y + YA = -X$, and put $-u = Ha + \frac{1}{2}b$. We obtain

$$\rho H = \mathfrak{A}(uu^*) + \tfrac{1}{2}\rho\, \mathfrak{A}(G), \tag{5}$$

$$\mathfrak{A}(uu^*)a + \rho u + \tfrac{1}{2}\rho(b + c) = 0, \tag{6}$$

where $c = \mathfrak{A}(G)a$. If the quadratic vector equation (6) has a real solution, then from formula (5) we find a matrix $H > 0$ satisfying inequality (4). Equation (6) can be written in the form

$$
\begin{aligned}
A^*U + UA &= -uu^*, \quad &\text{(I)} \\
Ua + \rho u + \tfrac{1}{2}\rho(b + c) &= 0, \quad &\text{(II)}
\end{aligned}
\tag{7}
$$

where the vector c is determined from the equations

$$
\begin{aligned}
A^*T + TA &= -G < 0, \quad &\text{(I)} \\
c &= Ta. \quad &\text{(II)}
\end{aligned}
\tag{8}
$$

We arrive at the theorem:

Theorem 1. *If, for some vector c determined by formulas (8), where $G > 0$ is some positive-definite symmetric matrix, the "resolving equation" (6) (or equation (7), (II)) has a real solution u, then the trivial solution of system (1) is stable in the large.*

It can be shown that all the computations connected with checking the conditions of the theorem can be carried out in a finite number of steps.

In equations (6) and (7) one may put $c = 0$ and require solvability of these equations not only for the given vector b, but also for all vectors b sufficiently close to the given one. This corresponds to the choice $G = \varepsilon E$, where $\varepsilon > 0$ is a sufficiently small number.

We obtain the resolving equations of A. I. Lur' e if we suppose that the matrix A is reducible to diagonal form, take the corresponding basis, and put $c = 0$.

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* This can easily be proved under certain additional natural assumptions on the function $\varphi(\sigma)$, which apparently were implicit in ([1]). If

$$
\int_0^{\pm\infty} \varphi(\sigma)\, d\sigma = \infty,
$$

this follows from (2). We note that, in order not to exclude the case encountered in applications where $\varphi'(\sigma) = \infty$ at some points, we do not require uniqueness of the solution of system (1). Moreover, stability in the large presupposes continuability of all solutions as $t \to \infty$, which under the conditions indicated in the text is proved.

** We write $G > 0$ if $(Gz, z) > 0$ for $z \neq 0$.

It is easy to derive, as in ([1]), that the condition $\Gamma^2 = \rho + (b, A^{-1}a) > 0$ is a necessary condition for system (7) to have real solutions. It can also be shown that, for the corresponding linear system with $\varphi(\sigma) = \mu\sigma$, the expansion of the root of the characteristic equation that tends to zero as $\mu = 0$ has the form $\lambda_0 = -\mu\Gamma^2(1 + \cdots)$, i.e., the condition $\Gamma^2 > 0$ is also a necessary condition for stability in the large.

From equations (7) one easily obtains the relation* $(u, A^{-1}a) = \rho + \sqrt{\rho}\Gamma$, using which one can lower the order of the determining system of equations by one.

Theorem 2. *Let, in system (1), $Aa = -\alpha a$, $\alpha > 0$. In order that the trivial solution of system (1) be stable in the large for every continuous function $\varphi(\sigma)$ satisfying conditions (2), it is necessary and sufficient that*

$$\Gamma^2 = \rho - \frac{1}{\alpha}(b, a) > 0.$$

In this case the quadratic equation (6) can be solved explicitly; one of the solutions will be:

$$u = -\frac{1}{2}\left[A^* - \frac{\alpha\Gamma}{\sqrt{\rho}}E\right]^{-1}(A^* - \alpha E)(b + c).$$

In the scalar case $n = 1$, equation (6) becomes an ordinary quadratic equation; this case is also a special case of Theorem 2. The case $n = 1$ is treated somewhat differently in ([4,5]), where nonlinear systems of another type were also considered.

Let $n = 2$. Suppose that $Aa \neq -\alpha a$, i.e., that the vectors $a_0 = a$ and $a_1 = A^{-1}a$ are linearly independent. Making in (1) the substitution $x = Sx_1$, $S = \|a_0|a_1\|$, i.e., passing to the basis a_0, a_1, we arrive at a system of the same form, where

$$A = \begin{pmatrix} -\alpha_0 & 1 \\ -\alpha_1 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}, \quad \alpha_0 > 0, \quad \alpha_1 > 0. \tag{9}$$

Carrying out the very simple calculations in this case for $c = 0$, we obtain:

Theorem 3. *In order that the trivial solution of system (1), where A, a, b have the form (9), be stable in the large, it is sufficient that*

$$\Gamma^2 = \rho + \beta_1 > 0, \quad \alpha_0(\rho\alpha_0 - \beta_0) > \alpha_1(\sqrt{\rho} - \Gamma)^2. \tag{10}$$

The elements η_0 and η_1 of the vector c in this case satisfy the inequalities

$$-\frac{\alpha_0}{\alpha_1}\eta_0 < \eta_1 < 0.$$

Consideration of equation (6) with a vector $c \neq 0$ gives nothing new. Therefore conditions (10) include all sufficient conditions that can be obtained by means of a Lyapunov function of the form (3).**

3°. With almost no changes one can consider the case where the characteristic equation of the open-loop system has one zero root. The determining equations (7) remain the same. Their number can be lowered by one.

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CITED LITERATURE

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2. E. B. Barbashin, N. N. Krasovskii, DAN, **76**, No. 3 (1952).

3. F. R. Gantmakher, *Matrix Theory*, 1954.

4. N. P. Erugin, *Applied Mathematics and Mechanics*, **16**, issue 4 (1952).

5. I. G. Malkin, *Applied Mathematics and Mechanics*, **16**, issue 3 (1952).

* An analogous relation (5.2) in ([1]).

** In particular, they either coincide with or are broader than the corresponding conditions in ([1]). The author does not know whether conditions (10) are necessary for stability in the large. It can be shown that, for stability of the linear system with $\varphi(\sigma) = \mu\sigma$, for all $\mu > 0$, it is necessary and sufficient that either conditions (10) hold, or that $\Gamma^2 > 0$, $\rho\alpha_0 - \beta_0 > 0$, $\beta_1 \leq 0$. Thus, for $\beta_1 > 0$, conditions (10) are also necessary.

Note: Figure translations are in progress. See original paper for figures.