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Abstract

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SOME QUESTIONS OF INFINITESIMAL BENDINGS OF SURFACES

In the present work, which is a further development of some of the author' s earlier results ([1,2]), a number of basic questions in the theory of infinitesimal bendings are considered, questions which are also connected with the momentless theory of shells. Special attention is devoted to the consideration of connections compatible with infinitesimal bendings and possessing the properties of normal-type boundary-value problems.

With every surface we shall associate a shell of constant thickness, for which it (the surface) serves as the middle surface. In what follows, infinitesimal bendings of a surface will simply be called bendings, and a momentless stressed equilibrium state of a shell will be called a momentless stress. Speaking of the variation of a quantity A expressing some property of a surface, we shall have in mind its first variation δA, corresponding to some bending.

Under bendings, the displacement vector $\mathbf{U}$, hereafter called the bending vector, satisfies the equality: $d\mathbf{U}d\mathbf{r} = 0$ ($\mathbf{r}$ is the radius vector of the surface), by virtue of which $d\mathbf{U} = \mathbf{V} \times d\mathbf{r}$, where $\mathbf{V}$ is the so-called rotation vector. This means that, as a result of the bending, every elementary area element of the surface is displaced as a rigid body.

The derivative of the vector $\mathbf{V}$ in some direction tangent to the surface s ($|s| = 1$) is the stress vector $\mathbf{T}_{(l)}$ of the momentless stress acting on the area element with normal l: $d\mathbf{V} = \mathbf{T}_{(l)}\, ds$ ($l \perp s$; $l \times s = n$—the unit normal to the surface; n, l, s is a right-handed trihedron). This makes it possible to associate with every bending a quite definite momentless stress, and conversely. This correspondence is one-to-one (up to a trivial bending) for a simply connected surface. In the case of a multiply connected surface, however, if its boundary consists of $m+1$ simple piecewise-smooth curves $L_0, L_1, \dots, L_m$, whose union we denote by L, then the fulfillment of the equalities

$$\int_{L_j} \mathbf{T}_{(l)}\, ds = 0, \qquad \int_{L_j} \mathbf{r} \times \mathbf{T}_{(l)}\, ds = 0 \quad (j = 0, 1, \dots, m) \tag{1}$$

is a necessary and sufficient condition for the uniqueness of the bending and rotation vectors. These equalities mean that on each boundary contour the stresses must be statically equivalent to zero.

Let I $\equiv a_{\alpha\beta}\,dx^\alpha dx^\beta$, II $\equiv b_{\alpha\beta}\,dx^\alpha dx^\beta$ be the first and second fundamental quadratic forms of the surface with respect to an arbitrarily chosen system of coordinates x^1, x^2. Under bendings $\delta a_{\alpha\beta} = 0$, while $\delta b_{\alpha\beta}$ satisfy the following system of equations:

$$\nabla_\alpha T^{\alpha\beta} = 0 \quad (\beta = 1,2), \qquad b_{\alpha\beta} T^{\alpha\beta} = 0 \quad (T^{\alpha\beta} = T^{\beta\alpha}), \tag{2}$$

where $T^{\alpha\beta} = c^{\alpha\lambda} c^{\beta\gamma} \delta b_{\lambda\gamma}$ ($c^{11} = c^{22} = 0$, $c^{12} = -c^{21} = 1/\sqrt{a}$; $a = a_{11}a_{22} - a_{12}^2$) constitute the contravariant components of the stress tensor of the momentless stress: $\mathbf{T}_{(l)} = T^{\alpha\beta} l_\alpha \mathbf{r}_\beta$ ($l_\alpha = l\mathbf{r}_\alpha$). The components of the bending vector $\mathbf{U} = u_\alpha \mathbf{r}^\alpha + u_0 \mathbf{n} \equiv u^\alpha \mathbf{r}_\alpha + u_0 \mathbf{n}$ satisfy the system of equations adjoint to (2):

$$\frac{1}{2}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - b_{\alpha\beta} u_0 = 0 \quad (\alpha, \beta = 1,2). \tag{3}$$

Therefore the equality will hold

$$\int_{\dot{L}} T^{\alpha\beta} l_\alpha u_\beta \, ds \equiv \int_L \mathbf{U}\mathbf{T}_{(l)} \, ds = 0, \tag{4}$$

meaning that *the total work of the contour forces of a momentless stress on the displacements corresponding to any bending of the middle surface is zero*, i.e., these bendings are possible displacements with respect to constraints equivalent to the contour forces of a momentless stress.

If a thin elastic shell obeying Hooke' s law is considered, then with sufficient approximation one may assume that every bending of the middle surface will maintain in the shell a state of purely moment stress equilibrium, since such a deformation does not cause elongations of linear elements or shears. Therefore constraints compatible with bendings and uniquely determining them will be equivalent to static conditions ensuring in the elastic shell a purely moment stress state of equilibrium. In particular, under rigid constraints incompatible with bendings, an elastic shell cannot be in a state of purely moment (nonzero) stress equilibrium. But rigid constraints with a large margin of strength are in practice not always acceptable. In the presence of such constraints, any deformation of the surface will cause extensions (compressions) and shears, which may sometimes lead to undesirable consequences (formation of cracks and folds, destruction of constraints, etc.).

Therefore, of considerable interest are such rigid constraints as possess certain properties of correctness. We have in mind such rigid constraints which, without having a significant margin of strength, are comparatively easily susceptible to perturbations bringing them to constraints compatible with bendings. But the realization and maintenance of constraints of this kind require special geometric and mechanical devices ([2]).

In the case of a surface of positive curvature, constraints of this kind are, generally speaking, equivalent to kinematic constraints fixing at each boundary point only one degree of freedom.

A very important class of constraints of this kind is formed by linear constraints of the form $\mathbf{R}(\mathbf{U}) = f$, where $\mathbf{R}$ is a certain homogeneous additive operator satisfying the following conditions: the homogeneous constraints $\mathbf{R}(\mathbf{U}) = 0$ admit only a finite number ν of linearly independent bendings, while the inhomogeneous (perturbed) constraints $\mathbf{R}(\mathbf{U}) = f$, for any continuous right-hand side orthogonal to some system of μ linearly independent functions, are compatible with bendings of the surface. For $\nu = \mu = 0$ we shall have correct constraints. If $\nu > 0$, and $\mu = 0$, then constraints of the form $\mathbf{R}(\mathbf{U}) = f$ will be called quasi-correct. By adjoining to them specially chosen constraints (for example point constraints) restricting ν degrees of freedom, they are transformed into correct constraints. For $\nu \geqslant 0$ and $\mu > 0$, constraints of the form $\mathbf{R}(\mathbf{U}) = f$ will be called weakly correct. These constraints are more difficult to transform into correct constraints. Here it is required simultaneously both to add new constraints restricting a finite number ν of degrees of freedom and to release a certain number of existing constraints fixing a finite number μ of degrees of freedom.

Constraints of the form $\mathbf{R}(\mathbf{U}) = f$ are obtained, for example, when the contour of a surface slides (is in continuous contact) along an ideally smooth surface of a solid body ([2]). Below we shall see that constraints of this kind can also be obtained by gluing surfaces.

The tensor $\delta b_{\alpha\beta}$ can be expressed through two scalar functions p and q, having an immediate geometric meaning:

$$\frac{1}{2}(k_1 - k_2)\,\delta b_{\alpha\beta} = (H b_{\alpha\beta} - K a_{\alpha\beta})p + \frac{1}{2}(c_{\alpha\lambda} b^{\lambda}_{\beta} + c_{\beta\lambda} b^{\lambda}_{\alpha})q, \tag{5}$$

$$p = \frac{2}{k_1 - k_2}\,\delta H, \qquad q = (k_1 - k_2)\,\delta\chi, \tag{6}$$

where k_1, k_2 are the principal curvatures ($K = k_1 k_2$); $\delta H, \delta\chi$ are the variations of the mean curvature and of the angle of inclination of the principal directions, whose unit vectors we denote by $\mathbf{s}_1, \mathbf{s}_2$ (the trihedron $\mathbf{s}_1, \mathbf{s}_2, \mathbf{n} = \mathbf{s}_1 \times \mathbf{s}_2$ forms a right-handed system).

For the normal and tangential stresses on an element with normal $\mathbf{l}$ we have the formulas:

$$G_l = (k_1 \sin^2\varphi - k_2 \cos^2\varphi)p + \sin 2\varphi\, q, \qquad H_l = H \sin 2\varphi\, p + \cos 2\varphi\, q, \tag{7}$$

where φ is the angle of inclination of $\mathbf{l}$ to the principal direction $\mathbf{s}_1$. Obviously, $T_{(l)} = \mathbf{l}G_l + \mathbf{s}H_l$.

It follows from (7) that $G_l + G_s = 2\delta H$, i.e. the semisum of the normal stresses of a momentless stress state, applied to two mutually perpendicular elements of a normal section, is equal to the variation of the mean curvature.

The variations of the curvature k and torsion $\varkappa$ of a curve L lying on the surface and having tangential normal $\mathbf{l}$ are expressed by the formulas

$$\delta k = G_l \cos\theta, \qquad \delta\varkappa = H_l - \frac{d\delta\theta}{ds}, \qquad \delta\theta = -\frac{\sin\theta}{k} G_l, \tag{8}$$

where θ is the angle between the principal normal of L and the normal to the surface (the natural trihedron $\mathbf{s}, \mathbf{m}, \mathbf{b}$ of the curve L has the same orientation as the trihedron $\mathbf{s}, \mathbf{n}, \mathbf{l}$). It is not difficult to see that G_l and H_l are equal to the variations of the normal curvature and of the geodesic torsion of the surface in the direction of the curve L.

The variations of the unit vectors $\mathbf{n}, \mathbf{l}, \mathbf{s}, \mathbf{m}, \mathbf{b}$ are computed from the formulas:

$$\delta\mathbf{n} = \mathbf{V}\times\mathbf{n}, \quad \delta\mathbf{l} = \mathbf{V}\times\mathbf{l}, \quad \delta\mathbf{s} = \mathbf{V}\times\mathbf{s}, \quad \delta\mathbf{m} = \mathbf{V}\times\mathbf{m} - \delta\theta\,\mathbf{b}, \quad \delta\mathbf{b} = \mathbf{V}\times\mathbf{b} + \delta\theta\,\mathbf{m}. \tag{9}$$

If L is the line along which the surfaces S^+ and S^- are glued, then from formulas (9) and from the continuity of the deformation ($\mathbf{U}^+ = \mathbf{U}^-$ on L) it follows that the rotation vector $\mathbf{V}$, upon passing through the gluing line, undergoes a jump expressed by the formula

$$\mathbf{V}^+ - \mathbf{V}^- = -\delta\vartheta\,\mathbf{s}, \qquad \delta\vartheta = -\frac{k\sin\vartheta}{k^+ k^-}\,\delta k, \qquad \vartheta = \theta^- - \theta^+, \tag{10}$$

where ϑ is the gluing angle ($\cos\vartheta = \mathbf{n}^+\mathbf{n}^-$); k^+ and k^- are the normal curvatures of the surfaces S^+ and S^- in the direction of the curve L. Differentiating both sides of equality $(10)_1$ along the arc, we obtain the following compatibility conditions:

$$G_{l+}\cos\theta^+ - G_{l-}\cos\theta^- = 0, \qquad H_{l+} - H_{l-} + \frac{d\delta\vartheta}{ds} = 0. \tag{11}$$

Suppose that the tangent to the gluing line coincides at every point with an asymptotic direction of the surface S^-. Then $\cos\theta^- = 0$ and $\delta k = 0$ along L. If, moreover, the tangent to L nowhere coincides with an asymptotic direction of the surface S^+, then $\cos\theta^+ \neq 0$ and $G_{l+} = 0$ on L.

We shall say that the surfaces are glued rigidly if the angle ϑ satisfies the condition $\delta\vartheta = 0$ (the gluing angle does not change). In this case $G_{l+} = 0$ and $G_{l-} = 0$ along L (therefore, $\delta k = 0$ and $H_{l+} = H_{l-}$). Let us note that if $\vartheta = 0$ or $\vartheta = \pi$, then the condition $\delta\vartheta = 0$ is always satisfied.

Thus, a boundary condition equivalent to the condition $\delta k = 0$ (for $\cos\theta \neq 0$) can be realized either by rigid gluing, or by gluing the surface along its contour to a surface for which this contour is an asymptotic curve (or the envelope of such curves).

Let us consider a regular surface of positive curvature. Suppose it is referred to a conjugate isometric coordinate system. Then

$$II = \sqrt{aK}\,(dx^2 + dy^2)$$

(K is the total curvature), and the systems of equations (2); (3) can be written in complex form:

$$\frac{\partial w}{\partial \bar{z}} + B\overline{w} = 0, \qquad \frac{\partial w_*}{\partial z} - \overline{B} w_* = 0 \qquad \left(\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right), \tag{12}$$

where

$$w_* = a\sqrt[4]{K}\,(T^{11} - iT^{12}) \quad (T^{22} = -T^{11}), \qquad w = (a\sqrt[4]{K})^{-1/2}(u_1 + iu_2), \tag{13}$$

$$B = -(\mathbf{r}^1 + i\mathbf{r}^2)\frac{\partial^2 \mathbf{r}}{\partial z^2} = \frac{1}{4}(\Gamma^1_{22} - \Gamma^1_{11} + 2\Gamma^2_{12}) + \frac{i}{4}(\Gamma^2_{22} - \Gamma^2_{11} - 2\Gamma^1_{12}). \tag{14}$$

The equations (12) reduce to the Cauchy–Riemann equation ($w_{\bar{z}} = 0$) only in the case when $B = 0$. This holds only for those surfaces which, as follows easily from equality (14) and from the condition $\mathbf{n}\mathbf{r}_{\bar{z}\bar{z}} = 0$, satisfy an equation of the form

$$\frac{\partial^2 \mathbf{r}}{\partial z^2} + (f(x,y) + ig(x,y))(\mathbf{r}_1 + i\mathbf{r}_2) = 0. \tag{15}$$

Let the surface S under consideration be an ovaloid with holes whose contours are piecewise smooth curves $L_0, L_1, \dots, L_m$ (their totality will be denoted by L). Let these contours be glued to certain regular surfaces $S_0, S_1, \dots, S_m$, respectively. Denote by S_* the piecewise smooth surface $S + S_0 + S_1 \dots + S_m$, assuming that the gluing has been carried out rigidly in those parts of L which do not touch the asymptotic directions of the surfaces $S_0, S_1, \dots, S_m$.

If $S_0, S_1, \dots, S_m$ are narrow strips, then the surface S_* will be called an ovaloid with rimmed edges. Since on an ovaloid $\cos\theta \neq 0$, under bendings of S_* along L we shall have the boundary condition $G_l = 0$ (for S). In view of equality (4), the homogeneous contour condition conjugate to $G_l = f$ will be the kinematic condition $u_s = 0$. Thus, for equations (12) we have conjugate homogeneous boundary-value problems $G_l = 0$ and $u_s = 0$, studied in the author' s work (^{1}) (§§ 8 and 9). Their indices are respectively equal to: $n = 2m - 2$, $n^* = 1 - m$. For the case of an ovaloid with one hole ($m = 0$), $n = -2$. Therefore the homogeneous problem $G_l = 0$ has no nontrivial solutions. For $m > 1$ these problems always admit $3(m-1)$ linearly independent solutions. But they must satisfy the conditions (1) of uniqueness of the vectors $\mathbf{U}$ and $\mathbf{V}$. On this basis we conclude that also in the case $m > 1$ a (nontrivial) bending of the surface S is impossible. We arrive at an analogous result also for $m = 1$. Thus the following holds.

Theorem. *Under an infinitesimal bending of the piecewise smooth surface* $S_* = S + S_0 + \dots + S_m$*, the ovaloid* S *(with holes) remains rigid. In particular, an ovaloid with rimmed edges is rigid.*

The inhomogeneous boundary condition $G_l = f$ may be regarded as a perturbation of the rigid constraint $G_l = 0$ (for example, an inaccuracy in satisfying the condition $\delta\vartheta = 0$). But this problem is solvable only when $3m+3$ conditions of the form

$$\int_l f X_j \, ds = 0 \quad (j = 1, \dots, 3m+3)$$

are fulfilled. Therefore the rigid constraint $G_l = 0$ is weakly correct.

It is also of interest to determine under what conditions the surfaces $S_0, \dots, S_m$ will also be rigid. For example, if S_j is a surface of positive curvature, then it will certainly be rigid. In this case, on its contour L_j, G_l and H_l vanish, and this is sufficient to have $p \equiv q \equiv 0$. If, however, L_j contains an arc of an asymptotic line of the surface S_j, then S_j may fail to be rigid. If the holes are plane curves, then we shall have a truncated ovaloid, whose rigidity was proved by another method in work ([3]). In this case $S_0, \dots, S_m$ are plane figures, which always admit a bending.

By gluing surfaces one can also obtain a number of other boundary conditions for equations (12). For example, if two simply connected pieces of ovaloids are glued along their common contour L, then the conjugacy conditions (11) lead to a generalized formulation of the Hilbert problem ([4]) for equations (12).

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REFERENCES

1. I. N. Vekua, *Matem. sborn.*, **31** (73): 2, 216 (1952).

2. I. N. Vekua, *Czechoslovak Math. J.*, **6** (81), 143 (1956).

3. I. V. Efimov, *Uspekhi Mat. Nauk*, **3**, 2(24), 47 (1948).

4. N. I. Muskhelishvili, *Singular Integral Equations*, Moscow, 1946.

Note: Figure translations are in progress. See original paper for figures.