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Abstract

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ON EQUATIONS WITH OPERATORS FORMING AN ACUTE ANGLE

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One of the frequently used methods for studying complicated equations consists in comparing these equations with simpler ones whose properties are known. In this case, from the "closeness," in a certain sense, of the equations there follows the commonality of a number of important properties (solvability of equations, the possibility of applying approximate methods, etc.).

In the present article a simple principle of comparison is proposed, which makes it possible to obtain a number of new propositions on elliptic and parabolic equations.

1. We shall say that linear operators A and B, acting in a Hilbert space H, form an acute angle if they have a common domain of definition $D = D(A) = D(B)$, vanish only at zero, and if

$$(Ax, Bx) \geqslant m\|Ax\| \cdot \|Bx\| \qquad (x \in D), \tag{1}$$

where $m > 0$. Obviously, $m \leqslant 1$.

An elementary calculation shows that the gap, in the sense of M. A. Krasnosel' skii and M. G. Krein [1], between the linear sets $R(A)$ and $R(B)$ of values of the operators A and B, which form an acute angle, does not exceed $1 - m$. Hence follows Theorem 1.

Theorem 1. *Let the operators A and B form an acute angle. Then they have identical defect indices, i.e. the orthogonal complements of $\overline{R(A)}$ and $\overline{R(B)}$ have the same dimension.*

2. Let $\{R_n\}$ be a monotonically increasing sequence of subspaces of the space H, and

$$\overline{\bigcup_{n=1}^{\infty} R_n} = H.$$

Denote by P_n the operator of orthogonal projection onto R_n. To obtain an approximate solution of the equation $Bx = f$, one may seek in the subspaces R_n a solution x_n of the "approximate" equations $P_n Bx = P_n f$.

Theorem 2. *Let the operators A and B form an acute angle and $R(A) = R(B) = H$. Let the subspaces R_n be invariant with respect to the self-adjoint operator A.*

Then, for each n, the equation $P_n Bx = P_n f$ has a solution $x_n \in R_n$. As $n \to \infty$, the approximate solutions x_n converge in norm to the solution x of the equation $Bx = f$, and the norms of the residuals $Bx_n - f$ tend to zero.

Consider the problem

$$\frac{dx}{dt} + Bx = 0, \qquad x(0) = x_0 \qquad (x_0 \in D(B)). \tag{2}$$

As its approximate solution $x_n(t)$ we shall regard the solution of the problem

$$\frac{dx}{dt} + P_n Bx = 0, \qquad x_n(0) = P_n x_0. \tag{3}$$

Theorem 3. Let the conditions of Theorem 2 be satisfied, and suppose that for every $x \in D(B)$ the inequalities

$$(Ax, x) \geq (x, x), \qquad (Bx, x) \geq (x, x) \tag{4}$$

hold. Then problems (2) and (3) are solvable, and for every $T > 0$

$$\lim_{n\to\infty} \max_{[0,T]} \|x_n(t) - x(t)\| = 0; \tag{5}$$

$$\lim_{n\to\infty} \max_{[0,T]} \left\| \frac{dx_n(t)}{dt} + Bx_n(t) \right\| = 0, \tag{6}$$

where $x(t)$ is the solution of problem (2).

The proof of Theorem 3 uses one theorem of Yosida–Hille ([2]).

3. Let the operators A and B be normally solvable and form an acute angle. It is easy to see that the perturbed operators $A + F_1$ and $B + F_2$ also form an acute angle, provided the operators F_1 and F_2 are bounded and have sufficiently small norms. In some cases one can show that the acuteness of the angle is preserved also under unbounded perturbations.

Theorem 4. Let A and B be positive-definite self-adjoint operators with a common domain of definition D. Let the constants $\alpha_1 > 0$, $\alpha_2 > 0$, and $m > 0$ characterize the comparability of these operators and the angle between them:

$$\alpha_1 \|Ax\| \leq \|Bx\| \leq \alpha_2 \|Ax\|, \qquad (Ax, Bx) \geq m\|Ax\| \cdot \|Bx\| \quad (x \in D).$$

Let, finally, an operator F defined on D satisfy the inequality

$$\|Fx\|^2 \leq \delta^2 \|Ax\|^2 + c^2 \|A^{1-\varepsilon} x\|^2 \qquad (x \in D) \tag{7}$$

for some $c > 0$, $\varepsilon > 0$, and $0 < \delta < \alpha_1 m$.

Then, for sufficiently large k, the operator $B + F + kI$ is normally solvable and forms an acute angle with the operator A.

In particular, the assertion of the theorem is true for the operator $A + F + kI$, if $0 < \delta < 1$. Therefore, if the operator A has a completely continuous inverse, then $A + F$ is an operator of Fredholm type.

4. Denote by H_1 the Hilbert space of functions integrable in the Bochner sense on $[0, T]$ with values in H. The scalar product in H_1 is defined by the equality

$$[x(t), y(t)]_{H_1} = \int_0^T (x(t), y(t))\, dt.$$

Let A be a positive-definite self-adjoint operator with domain D. The operator $d/dt + A$ is defined directly on those continuously differentiable functions $x(t) \in H_1$ for which the functions $Ax(t)$ are continuous and $x(0) = 0$. The closure of this operator in H_1 will also be denoted by $d/dt + A$ (one can show that this closure exists). The domain of definition of the constructed closed operator will be denoted by S.

It follows from known theorems (see, for example, ([3,4])) that the equation considered in H_1

$$\frac{dx}{dt} + Ax = f(t) \tag{8}$$

has a solution $x(t) \in S$, if $f(t) \in H_1$.

Theorem 5. Let the operators $A(t)$, $B(t)$, and $F(t)$, for every $t \in [0, T]$, be defined on D and satisfy the conditions of Theorem 4 with constants independent of t, $\alpha_1 > 0$, $\alpha_2 > 0$, $m > 0$, $c > 0$, $\varepsilon > 0$, and

$$0 < \delta < 2\sqrt{\alpha_1 m + 1} - 2.$$

Let, for every $x \in D$, the functions $A(t)x$,

$B(t)x$ are continuous, piecewise continuously differentiable, and the function $F(t)x$ is continuous in t. Then, for sufficiently large k, the operator $d/dt + B(t) + F(t) + kI$ acting in H_1, defined on S, is normally solvable and forms an acute angle with the operator $d/dt + A(t)$.

In particular, the assertion of the theorem is true for the operator $d/dt + A(t) + F(t) + kI$, if $0 < \delta < 4/5$.

5. We proceed to the consideration of concrete differential operators.

By $\overline{\Omega}$ we shall denote a closed bounded domain of n-dimensional space admitting a twice continuously differentiable mapping onto the unit ball with nonzero Jacobian. Let $\overset{0}{W}{}_2^2$, as usual, denote the closure in the norm

$$\|u\|^2_{\overset{0}{W}{}_2^2} = \int_\Omega \left[u^2 + \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 + \sum_{i,k=1}^n \left(\frac{\partial^2 u}{\partial x_i \partial x_k}\right)^2 \right] dx \tag{9}$$

of the set of functions three times continuously differentiable in $\overline{\Omega}$ and vanishing on the boundary. In $L_2(\Omega)$ consider the differential operators

$$A \equiv -\sum_{i,k=1}^n a_{ik}(x) \frac{\partial^2}{\partial x_i \partial x_k} + \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i} + a(x) I, \tag{10}$$

$$B \equiv -\sum_{i,k=1}^n b_{ik}(x) \frac{\partial^2}{\partial x_i \partial x_k} + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} + b(x) I \tag{11}$$

with domain of definition $\overset{0}{W}{}_2^2$. We shall assume that on $\overline{\Omega}$ the functions a_{ik} and b_{ik} are continuously differentiable, while the functions a_i, b_i, a, and b are continuous, and that the forms $\sum_{i,k=1}^n a_{ik}\xi_i\xi_k$ and $\sum_{i,k=1}^n b_{ik}\xi_i\xi_k$ are positive definite for every $x \in \overline{\Omega}$.

Theorem 6. There exist numbers $m > 0$, $n > 0$ such that for all functions $u(x) \in \overset{0}{W}{}_2^2$ the inequality

$$\int_\Omega Au \cdot Bu \, dx \geq m \int_\Omega \sum_{i,k=1}^n \left(\frac{\partial^2 u}{\partial x_i \partial x_k}\right)^2 dx - n \int_\Omega \left[u^2 + \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 \right] dx. \tag{12}$$

holds.

If the function $a(x)$ is bounded below by a sufficiently large number depending only on the functions a_{ik} and a_i, then for $A = B$ there follows from (12) the known inequality of O. A. Ladyzhenskaya ([6])

$$\|Au\|^2_{L_2} \geq c^2 \|u\|^2_{\overset{0}{W}{}_2^2} \quad (u \in \overset{0}{W}{}_2^2).$$

Under the same condition, the operator A forms an acute angle with the operator $B + kI$, where k is a sufficiently large number.

6. Let $H = L_2(\Omega)$ and let Δ be the Laplace operator defined on $\overset{0}{W}{}_2^2$. From the self-adjoint positive-definite operator $-\Delta$ we construct the operator

$d/dt - \Delta$ acting in H_1 (see item 4). We denote its domain of definition by S. From Theorem 5 there follows the following theorem.

Theorem 7. Let the operator

$$A(t) \equiv - \sum_{i,k=1}^{n} a_{ik}(t,x) \frac{\partial^2}{\partial x_i \partial x_k} + \sum_{i=1}^{\infty} a_i(t,x) \frac{\partial}{\partial x_i} + a(t,x) I$$

for each $t \in [0, T]$ satisfy the conditions of item 5. Let the functions

$a_{ik}(t,x)$ are continuous, piecewise continuously differentiable, and the functions $a_i(t,x)$ and $a(t,x)$ are continuous in t. Then, for sufficiently large k, the operator $d/dt + A(t) + kI$ acting in H_1 is normally solvable and forms an acute angle with the operator $d/dt - \Delta$.

7. Applying Theorem 1 to the operators considered in §§ 4–6, we obtain assertions on the solvability of various equations. We give examples.

If the operators $A(t)$ and $F(t)$ satisfy the conditions of § 4, then from the results of ([4]) it follows that the equation

$$\frac{dx}{dt} + A(t)x + F(t)x = f(t) \tag{13}$$

has a solution $x(t) \in S$, if $f(t) \in H_1$. This assertion remains valid also under weaker restrictions on the operator $A(t)$ than piecewise differentiability. It is enough to require that the operator $A(t)A^{-1}(0)$ be continuous in the operator norm. This is proved by approximating the operator $A(t)$ by specially chosen continuous, piecewise continuously differentiable operators. In particular, from this follows the solvability of parabolic-type equations with a second-order elliptic operator with zero boundary conditions, if the coefficients of this operator depend continuously on t. The assertions given generalize some theorems of O. A. Ladyzhenskaya from ([5,7]). The theorem of O. A. Ladyzhenskaya ([6]) on the solvability of the first boundary-value problem for the elliptic equation $Au = f$ follows from the results of § 5.

8. Application of Theorems 2 and 3 to the differential operators considered above (§ 5) gives theorems on the convergence of the Bubnov–Galerkin method for elliptic and parabolic equations. In addition to the convergence of the approximations to the solution, the residuals also converge to zero. This fact for self-adjoint elliptic equations was previously established by S. G. Mikhlin ([8]). Denote by P_n the operator of orthogonal projection onto the linear span R_n of the first n elements of the basis $\{e_n\}$, consisting of eigenvectors of some self-adjoint elliptic operator.

Theorem 8. Let the operator A satisfy the conditions of § 5. Then for each n there exists a solution $u_n \in R_n$ of the equation $P_{nAu}n = P_{nf}$. As $n \to \infty$, the solution u_n converges in norm to the solution u of the equation $Au = f$, and the norms of the residuals $Au_n - f$ tend to zero.

Theorem 9. Let the operator A satisfy the conditions of § 5. Then for each n there exists a solution $u_n(t)$ of the problem

$$\frac{du}{dt} + P_{nAu} = 0, \qquad u_n(0) = P_{nu}0 \quad (u_0 \in \mathring{W}_2^2).$$

As $n \to \infty$, uniformly in $t \in [0, T]$, the solution $u_n(t)$ converges in norm to the solution $u(t)$ of the problem

$$\frac{du}{dt} + Au = 0, \qquad u(0) = u_0,$$

and the norms of the residuals $du_n/dt + Au_n$ tend to zero.

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Note: Figure translations are in progress. See original paper for figures.