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1957

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Abstract

Full Text

MATHEMATICS

M. D. KHRIPTUN

ON A CLASS OF ENTIRE FUNCTIONS

(Presented by Academician V. I. Smirnov, 1 XI 1956)

It can be shown (^{3,4}) that the function

$$U_0^{(1)}\left(3\sqrt[3]{s}\right) = \sum_{n=0}^{\infty} \frac{s^n}{(2n)!\,n!} \tag{1}$$

in the Cauchy problem for an ordinary linear differential operator L of the 2nd order plays the same role as, for self-adjoint operators, is played by the function e^s, and for differential operators of the 1st order by the Bessel function

$$I_0\left(2\sqrt{s}\right) = \sum_{n=0}^{\infty} \frac{s^n}{n!\,n!},$$

namely: for $s = zu^2$ the function (1) will be the kernel of the integral representation of the operator e^{zA}, where $A = L_0^{-1}$ is the operator inverse to L for zero initial data (at the point $x = 0$).

Making in (1) the substitution $s = (z/3)^3$ (z a complex variable), replacing $n!$ by $\Gamma(n+p+1)$, and multiplying by $(z/3)^p$, we obtain the function

$$U_p^{(1)}(z) = \sum_{n=0}^{\infty} \frac{(z/3)^{p+3n}}{(2n)!\,\Gamma(n+p+1)}, \tag{2}$$

which satisfies the equation

$$\begin{aligned} z^3U'''(z) + {}^3/_2\, z^2U''(z) - {}^1/_2\, z(6p^2 + 3p + 1)U'(z) + \\ + (2p^3 + 3p^2 - {}^1/_4 z^3)U(z) = 0. \end{aligned} \tag{3}$$

It turned out that its solutions possess properties very similar to the properties of cylindrical functions. The purpose of the present note is to give a brief account of these properties.

§ 1. The given equation has two singular points: a regular one $z = 0$, and an irregular one $z = \infty$ (^{1}), p. 491).

Applying the usual arguments, we obtain:

I. If $p \neq k/3,\ p \neq (2k-3)/6$, where $k = 0,\ \pm 1,\ \pm 2, \ldots$, then three independent solutions of equation (3) are as follows: the first has the form given by formula (2), while the other two take the form

$$U^{(2)}_{p+{}^1/_2}(z) = \sum_{n=0}^{\infty} \frac{(z/3)^{p+{}^3/_2+3n}}{(2n+1)!\,\Gamma(n+p+{}^3/_2)}, \tag{4}$$

$$U^{(3)}_{-2p}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(2p-2n)(z/3)^{-2p+3n}}{\Gamma(2p)\,n!}. \tag{5}$$

We shall call these solutions, respectively, a solution of the first kind of the first type with index p; of the first kind of the second type with index $p + \frac{3}{2}$; of the first kind of the third type with index $-2p$.

II. Let $p = k/3$.

a) $k = 0, 1, 2, \ldots$. The solutions corresponding to the indices p and $p + \frac{3}{2}$ remain, in form, the same as (2), (4). The solution corresponding to the index $-2p$ is constructed by the general Frobenius method (([1]), pp. 534–545):

$$M^{(1)}_{-2p}(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!\,\Gamma(n+p+1)} \{3\ln z - \psi(n) - \psi(p+n) - \psi(n-\tfrac{1}{2})\} \left(\frac{z}{3}\right)^{p+3n} +$$

$$+ \sum_{n=0}^{p-1} \frac{\Gamma(2p-2n)}{\Gamma(2p)n!} \left(\frac{z}{3}\right)^{-2p+3n}, \tag{6}$$

where

$$\psi(\tau) = \left[\frac{d}{dt}\log\Gamma(t+1)\right]_{t=\tau}.$$

We shall call this solution a solution of the second kind of the first type with index $-2p$.

b) If $k = -1, -2, -3, \ldots$, then the form of the solutions corresponding to the indices $-2p$ and $p + \frac{3}{2}$ remains the same as in case I, while the solution corresponding to the index p is constructed in the same way as $M^{(1)}_{-2p}(z)$; it has the form:

$$M_p^{(3)}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(2p-2n)}{\Gamma(2p)n!} \{3\ln z - \psi(n) - \psi(-p+n) - \psi(-p-\tfrac{1}{2}+n)\} \left(\frac{z}{3}\right)^{-2p+3n} +$$

$$+ \sum_{n=0}^{-p-1} \frac{1}{(2n)!\,\Gamma(n+p+1)} \left(\frac{z}{3}\right)^{p+3n}; \qquad (7)$$

we shall call it a solution of the second kind of the third type with index p.

III. Let $p = (2k-3)/6$.

a) $k = 2, 3, 4, \ldots$. The solutions corresponding to the indices $p + \frac{3}{2}$ and p have the form (4) and (2), while the solution corresponding to the index $-2p$ takes the form

$$M_{-2p}^{(2)}(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!\,\Gamma(n+p+\frac{3}{2})} \{3\ln z - \psi(n) - \psi(n+\tfrac{1}{2}) -$$

$$-\psi(p+n+\tfrac{1}{2})\} \left(\frac{z}{3}\right)^{p+\frac{3}{2}+3n} + \sum_{n=0}^{p+\frac{1}{2}-1} \frac{\Gamma(2p-2n)}{\Gamma(2p)n!} \left(\frac{z}{3}\right)^{-2p+3n}; \qquad (8)$$

we call it a solution of the second kind of the second type with index $-2p$.

b) $k = 0, \pm 1, -2, -3, \ldots$. The solution corresponding to the index $p + \frac{3}{2}$ has the form

$$M_{p+\frac{3}{2}}^{(3)}(z) =$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(2p-2n)}{\Gamma(2p)n!} \{3\ln z - \psi(n) - \psi(-p+n) - \psi(-p-\tfrac{1}{2}+n)\} \left(\frac{z}{3}\right)^{-2p+3n} +$$

$$+ \sum_{n=0}^{-p-\frac{1}{2}-1} \frac{1}{(2n+1)!\,\Gamma(n+p+\frac{3}{2})} \left(\frac{z}{3}\right)^{p+\frac{3}{2}+3n}, \qquad (9)$$

we call it a solution of the second kind of the third type with index $p + \frac{3}{2}$, and the remaining two solutions have the form (2) and (5), respectively.

IV. Let us single out the case $p = 0$; then all three independent solutions take the form:

$$U_0^{(1)}(z) = \sum_{n=0}^{\infty} \frac{(z/3)^{3n}}{(2n)!n!},$$

$$M_0^{(1)}(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!\,n!} \left\{ \ln z - \frac{2}{3}\psi(n) - \frac{1}{3}\psi\left(n - \frac{1}{2}\right) \right\} \left(\frac{z}{3}\right)^{3n},$$

$$U_{1/2}^{(2)}(z) = \sum_{n=0}^{\infty} \frac{(z/3)^{1/2+3n}}{(2n+1)!\,\Gamma(n+3/2)}.$$

In all these formulas

$$\psi(r) = \left[\frac{d}{dt} \log \Gamma(t+1)\right]_{t=r}.$$

Solutions of the first and second kinds of equation (3) are analogous to Bessel functions of the first and second kinds.

§ 2. **The generating function and integral representation for integer indices.** It is easy to verify the relation:

$$e^{zt/3} \operatorname{ch} \frac{z}{3\sqrt{t}} = \sum_{p=-\infty}^{\infty} t^p U_p^{(1)}(z), \tag{10}$$

where in formula (2) the terms corresponding to the poles of the function $\Gamma(n+p+1)$ are absent. Hence

$$U_p^{(1)}(z) = \frac{1}{2\pi i} \oint_{(0)} t^{-p-1} e^{zt/3} \operatorname{ch} \frac{z}{3\sqrt{t}}\, dt \quad (p = 0, \pm1, \pm2, \ldots). \tag{11}$$

Similarly

$$\sqrt{t}\, e^{zt/3} \operatorname{sh} \frac{z}{3\sqrt{t}} = \sum_{p=-\infty}^{\infty} t^p U_{p+1/2}^{(2)}(z). \tag{12}$$

Hence

$$U_{p+1}^{(2)}(z) = \frac{1}{2\pi i} \oint_{(0)} t^{-p-1} \sqrt{t}\, e^{zt/3} \operatorname{sh} \frac{z}{3\sqrt{t}}\, dt. \tag{13}$$

§ 3. **Integral representations of the function U with an arbitrary index.** Substituting the known expression

$$\frac{1}{\Gamma(n+p+1)} = \frac{1}{2\pi i} \int_{l'} e^{\tau} \tau^{-(n+p+1)}\, d\tau \tag{14}$$

((2), p. 275) into (2), after transformations we obtain

$$U_p^{(1)}(z) = \frac{1}{2\pi i} \int_{l'} \left(\frac{z}{3}\right)^p \tau^{-p-1} e^{\tau} \operatorname{ch} \left(\frac{z}{3}\sqrt{\frac{z}{3\tau}}\right) d\tau.$$

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Fig. 1

Figure 1: Fig. 1

Assuming that

$$|\arg z| < \pi/2 \tag{15}$$

and putting $\tau = zt/3$, we obtain

$$U_p^{(1)}(z) = \frac{1}{2\pi i}\int_l t^{-p-1} e^{zt/3}\,\mathrm{ch}\,\frac{z}{3\sqrt{t}}\,dt, \tag{16}$$

where as the path of integration one may take the former contour l' (on the basis of condition (15)).

In the same way we obtain two other representations:

$$U_{p+1/2}^{(2)}(z) = \frac{1}{2\pi i}\int_l t^{-p-1} e^{zt/3}\,\mathrm{sh}\,\frac{z}{3\sqrt{t}}\,dt, \tag{17}$$

$$U_{-2p}^{(3)}(z) = \frac{1}{2\pi i}\int_l t^{2p-1} e^{\frac{z}{3}\left(\frac{1}{t^2}+t\right)}\,dt. \tag{18}$$

Introducing $X_1(z) = U_p^{(1)}(z) + U_{p+1/2}^{(2)}(z)$ and $X_2(z) = U_p^{(1)}(z) - U_{p+1/2}^{(2)}(z)$, putting $u = 1/\sqrt{t}$ for $X_2(z)$, $u = -1/\sqrt{t}$ for $X_2(z)$, we bring their representa-

···to the form (18); now replacing $u = e^{i\omega}$ and returning to the functions U, we obtain the representations

$$U_p^{(1)}(z) = \frac{1}{4\pi}\int_C \exp\left[\frac{z}{3}\left(e^{-2i\omega} + e^{i\omega}\right) + 2pi\omega\right] d\omega, \tag{19}$$

$$U_{p+\frac{1}{2}}^{(2)}(z) = \frac{1}{4\pi}\int_{C'} \exp\left[\frac{z}{3}\left(e^{-2i\omega} + e^{i\omega}\right) + 2pi\omega\right] d\omega, \tag{20}$$

$$U_{-2p}^{(3)}(z) = \frac{1}{2\pi}\int_{C''} \exp\left[\frac{z}{3}\left(e^{-2i\omega} + e^{i\omega}\right) + 2pi\omega\right] d\omega \tag{21}$$

through one and the same function over different contours, shown respectively in Fig. 1 a, b, and c.

Fig. 1

§ 4. **Addition theorems.** If

$$R = \sqrt[3]{(r_1 e^{-i\theta/2} - r_2)^2 (r_1 e^{i\theta} - r_2)},$$

where r_1, r_2 are complex variables, then

$$U_0^{(1)}(\lambda R) = \sum_{m=-\infty}^{\infty} U_m^{(1)}(\lambda r_1) U_{-m}^{(1)}(-\lambda r_2) e^{im\theta} +$$

$$+ \sum_{m=-\infty}^{\infty} U_{m+1}^{(2)}(\lambda r_1) U_{-m+2}^{(2)}(-\lambda r_2) e^{i\theta(m-\frac{1}{2})}. \tag{22}$$

For $\theta = 0$ a simpler formula is obtained:

$$U_0^{(1)}(\lambda(r_1 - r_2)) = \sum_{m=-\infty}^{\infty} U_m^{(1)}(\lambda r_1) U_{-m}^{(1)}(-\lambda r_2) +$$

$$+ \sum_{m=-\infty}^{\infty} U_{m+1}^{(2)}(\lambda r_1) U_{-m+2}^{(2)}(-\lambda r_2). \tag{23}$$

Formula (22) admits a generalization to the case of an arbitrary index ν:

$$U_\nu^{(1)}(\lambda R) e^{i\nu\psi} = \sum_{m=-\infty}^{\infty} U_m^{(1)}(\lambda r_1) U_{\nu-m}^{(1)}(-\lambda r_2) e^{im\theta} +$$

$$+ \sum_{m=-\infty}^{\infty} U_{m+1}^{(2)}(\lambda r_1) U_{\nu-m+2}^{(2)}(-\lambda r_2) e^{i\theta(m-\frac{1}{2})}, \tag{24}$$

where ψ must satisfy the relations: $R\cos\psi = r_1\cos\theta - r_2$; $R\sin\psi = r_1\sin\theta$, or $e^{i\psi} = (r_1 e^{i\theta} - r_2)/R$.

§ 5. There also hold differential recurrence relations.

In conclusion I express my sincere gratitude to the supervisor of the present work, M. K. Fage, for valuable instructions and assistance rendered to me in the course of carrying out the work.

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Received
29 X 1956

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Note: Figure translations are in progress. See original paper for figures.

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