E. E. FRADKIN and S. V. IZMAILOV

1.** Particles with arbitrary spins are described by equations of the form

1957

SovietRxiv

Abstract

Full Text

E. E. FRADKIN and S. V. IZMAILOV

ON ADMISSIBLE TRANSFORMATIONS OF EQUATIONS FOR PARTICLES WITH HIGHER SPINS

(Presented by Academician V. A. Fock, 29 XII 1956)

1. Particles with arbitrary spins are described by equations of the form

$$\left(L^i \frac{\partial}{\partial x_i} + i\chi\right) \Psi(x_0, x_1, x_2, x_3) = 0, \tag{1}$$

where $\Psi(x_0, x_1, x_2, x_3) \equiv \Psi(ct, x, y, z)$ is a wave function transforming according to a finite-dimensional representation of the full Lorentz group; L^i $(i = 0, 1, 2, 3)$ are square matrices; χ is a real constant different from zero.

The invariance of (1) under coordinate transformations of the full Lorentz group is ensured by the fulfillment of the conditions imposed on the matrices L^i:

$$[L^i I^{jk}] = L^i I^{jk} - I^{jk} L^i = g^{ij} L^k - g^{ik} L^j \quad (i, j, k = 0, 1, 2, 3); \tag{2}$$

$$[L^i T] = 0, \tag{3}$$

where I^{jk} are the infinitesimal transformations of the corresponding representation of the proper Lorentz group; T is the transformation corresponding to the reflection $x^0 \to x^0$, $x^\alpha \to -x^\alpha$ $(\alpha = 1, 2, 3)$. From (2), in particular, it follows that

$$L^\mu = [L^0 I^{0\mu}] \quad (\mu = 1, 2, 3). \tag{4}$$

The general theory of relativistically invariant equations of the form (1) has been studied in paper (1).

2. Let us consider possible linear transformations of the wave function $\Psi(x_0, x_1, x_2, x_3)$, i.e. transformations of the form $\Psi(x_0, x_1, x_2, x_3) = S\Psi'(x_0, x_1, x_2, x_3)$, which do not change the essential properties of equation (1), and establish the form of the matrix S for admissible transformations.

Admissible transformations include, first of all, transformations V that do not change the form of I^{jk} and T (1), i.e.

$$V^{-1} I^{jk} V = I^{jk}, \qquad V^{-1} T V = T \quad (j, k = 0, 1, 2, 3). \tag{5}$$

Since the equations of the form (1) considered by us may be obtained from the corresponding Lagrange function, an important role is played by the metric matrix A, by means of which the invariant Hermitian bilinear form $\Psi^* A \Psi^*$ is defined.

Thus, in order to relate the different representations of equation (1) for a particle with spin 3/2, given in papers (1-3), it is necessary that the transformation be able to change the absolute values of the eigenvalues of the metric matrix A. Whereas the matrices I^{jk}, T, and L^i transform by similarity, i.e., for example, $L^{i'} = S^{-1} L^{iS}$, the metric matrix A transforms congruently: $A' = S^* A S$**. Each of the matrices I^{jk}, T, and A can be represented[1] as a direct sum of matrices belonging to irreducible representations of the full Lorentz group. Therefore a transformation V satisfying conditions (5), in the representation in which A is diagonal, can be written in the form

$$V = \begin{vmatrix} v_1 I_1 & & & \\ & v_2 I_2 & & \\ & & \ddots & \\ & & & v_n I_n \end{vmatrix}, \tag{6}$$

where $v_1, v_2, \dots, v_n$ are arbitrary complex numbers.

The dimensions of the unit matrices I_i are determined by the dimensions of the corresponding irreducible representations of the full Lorentz group. Under such a transformation the eigenvalues of A corresponding to the i-th irreducible representation of the full Lorentz group are changed in the ratio $|v_i|^2$.

The essential properties of J^k, T, L^i, and A likewise do not change under arbitrary unitary transformations. In view of the Hermiticity of the metric matrix A, by means of some unitary transformation one can always pass to a representation in which A is diagonal, with the required order of arrangement of the eigenvalues. Therefore the general form of the admissible transformations will be*

---

* Here and below the asterisk denotes Hermitian conjugation.

** Under the transformation S the invariant Hermitian bilinear form $\Psi^* A L P \Psi$, where L is some matrix and P some operator, remains unchanged:

$$\Psi^* A L P \Psi = \Psi'^* S^* A L P S \Psi' = \Psi'^* A' L' P' \Psi',$$

where $A' = S^* A S,\ L' = S^{-1} L S,\ P' = S^{-1} P S.$

$$S = U_1 V U_2, \tag{7}$$

where U_1 and U_2 are arbitrary unitary transformations; V is a transformation of the form (6).

3. The wave function Ψ contains an arbitrary normalization factor; by disposing of it, we can make the eigenvalues of A belonging to the irreducible representation of the full Lorentz group with the maximum value of the spin equal to ± 1 in all equivalent representations. Then in formula (6) the corresponding $v_k = 1$, and the metric in the subspace with the maximum spin value remains unchanged under admissible transformations. Thus, for a particle of spin 1/2 the transformation of the wave function contains only one irreducible representation of the full Lorentz group, and the admissible transformations of the Dirac matrices reduce to arbitrary unitary transformations. The invariant of such transformations is the charge density $\Psi^* \Psi$.

For particles with higher half-integer spins $s = n + 1/2$ $(n = 1, 2, \dots, l)$, it is easy to show that the invariant of the admissible transformations (7) will be only the charge density of a resting free particle.

4. Let us consider in more detail the transformation of equation (1) for a particle of spin 3/2. In papers (1-3) various representations of the matrices L^i $(i = 0, 1, 2, 3)$ of equation (1) were given for a particle of spin 3/2 with positively defined charge; moreover, the Gupta representation (2) is directly connected with the equations and Lagrangian function of Pauli and Fierz (4) with auxiliary spinors. On the other hand, there is an analogous connection between the Petras representation (3) and the spin-tensor equations and the corresponding Lagrangian function given by Rarita and Schwinger (5) without additional quantities. Consequently, by indicating the connection between the Gelfand-Yaglom and Petras-Gupta representations, we shall thereby show the equivalence of the principal representations of the equation for a particle of spin 3/2 under our definition of admissible transformations (7). The passage from the Gelfand-Yaglom representation to the Gupta-Petras representations is effected by the transformations S_1 and S_2.

---

[*] Transformations (7) can also be extended to the Duffin-Kemmer equations for particles of spin 0 and 1. However, (7) violates the Hermiticity of the Kemmer matrices and the usual connection between the metric matrix and the Kemmer matrices.

$$S_1 = U_1 V_1 U_2, \qquad S_2 = U_1 V_2 U_3. \tag{8}$$

The connection between the Gupta and Petrash representations is effected by the transformation S_3:

$$S_3 = U_2^* V_3 U_3. \tag{9}$$

Here, for example, $L^{i'} = S_2^{-1} L_2^{iS}$, where L^i are the matrices in the Gelfand-Yaglom representation, and L^0 is written in the form:

$$L^0 = \left| \begin{matrix} c_{\tau\tau'}^{3/2} & & & & & \\ & c_{\tau\tau'}^{3/2} & & & & \\ & & c_{\tau\tau'}^{3/2} & & & \\ & & & c_{\tau\tau'}^{3/2} & & \\ & & & & c_{\tau\tau'}^{1/2} & \\ & & & & & c_{\tau\tau'}^{1/2} \end{matrix} \right|, \tag{10}$$

where $\|c_{\tau\tau'}^{3/2}\|$ and $\|c_{\tau\tau'}^{1/2}\|$ are the matrices found in work (1), formula (7.10); $L^{i'}$ are matrices written explicitly in Petrash' s representation.

The matrices U_i and V_i ($i = 1, 2, 3$) appearing in (8) and (9) have the form*:

$$V_i = \left| \begin{matrix} I & & & \\ & I & & \\ & & I & \\ & & & k_i I \end{matrix} \right|, \qquad \text{where} \quad I = \left| \begin{matrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{matrix} \right|; \tag{11}$$

$$i = 1, 2, 3; \qquad k_1 = \frac{1}{\sqrt{3}}, \qquad k_2 = 2 - \sqrt{3}, \qquad k_3 = 2\sqrt{3} - 3;$$

$$U_1 = \frac{1}{\sqrt{2}} \left| \begin{array}{ccccccccc} \bullet & \vert & \bullet & \bullet & \vert & \bullet & \vert & \bullet & \bullet \\ \bullet & -1 & \bullet & \bullet & \vert & \bullet & \vert & \bullet & \bullet \\ \hline \bullet & \vert & \bullet & \bullet & \vert & \bullet & \vert & \bullet & \bullet \\ \bullet & \vert & -1 & \bullet & \vert & \bullet & \vert & \bullet & \bullet \\ \hline \bullet & \vert & \bullet & -1 & \vert & \bullet & \vert & \bullet & \bullet \\ \bullet & \vert & \bullet & \bullet & -1 & \bullet & \vert & \bullet & \bullet \\ \hline \bullet & \vert & \bullet & \bullet & \vert & -1 & \vert & \bullet & \bullet \\ \bullet & \vert & \bullet & \bullet & \vert & \bullet & \vert & -1 & \bullet \end{array} \right| \tag{12}$$

$$U_2 = \left|\begin{array}{c|c|c|c}
u_1 & \begin{array}{cc}\bullet&\bullet\\ \bullet&\bullet\\ \bullet&\bullet\\ \bullet&\bullet\end{array} & u_1 & \begin{array}{cc}\bullet&\bullet\\ \bullet&\bullet\\ \bullet&\bullet\\ \bullet&\bullet\end{array} \\ \hline
u_1 & \begin{array}{cc}\bullet&\bullet\\ \bullet&\bullet\\ \bullet&\bullet\\ \bullet&\bullet\end{array} & -u_1 & \begin{array}{cc}\bullet&\bullet\\ \bullet&\bullet\\ \bullet&\bullet\\ \bullet&\bullet\end{array} \\ \hline
\begin{array}{cccc}\bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet\end{array} & \begin{array}{c}u_2\\ \\ -u_2\end{array} & \begin{array}{cccc}\bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet\end{array} & \begin{array}{c}u_2\\ \\ u_2\end{array}
\end{array}\right| \tag{13}$$

$$U_3 = \left|\begin{array}{c}
\begin{array}{c|c|c|c|c|c|c}
u_3 & \begin{array}{cccc}\bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet\end{array} & u_4 & \begin{array}{cc}\bullet&\bullet\\ \bullet&\bullet\\ \bullet&\bullet\end{array} & u_5 & \begin{array}{cc}\bullet&\bullet\\ \bullet&\bullet\\ \bullet&\bullet\end{array} & u_6
\end{array} \\ \hline
\begin{array}{c|c|c|c|c|c|c|c}
\begin{array}{cc}\bullet&\bullet\\ \bullet&\bullet\\ \bullet&\bullet\end{array} & -\,u_3 \quad -u_4 & \begin{array}{cc}\bullet&\bullet\\ \bullet&\bullet\\ \bullet&\bullet\end{array} & u_5 & \begin{array}{cc}\bullet&\bullet\\ \bullet&\bullet\\ \bullet&\bullet\end{array} & -u_6 & \begin{array}{cc}\bullet&\bullet\\ \bullet&\bullet\\ \bullet&\bullet\end{array}
\end{array} \\ \hline
\begin{array}{cccc}
u_7 & u_8 & u_9 & u_{10} \\
u_7 & u_8 & u_8 & u_{10}
\end{array}
\end{array}\right| \tag{14}$$

* The unitary matrices U_1, U_2, and U_3 reduce, respectively, the metric matrices A, A', and A'' in the Gelfand–Yaglom, Gupta, and Petrash representations to diagonal form, while the matrices V_1, V_2, and V_3 carry out the transition from one diagonal metric matrix to another, changing part of the eigenvalues.

where

$$u_1 = \left|\begin{array}{cccccc}
0 & -\frac{ia}{\sqrt{2}} & ia & ia & -\frac{ia}{\sqrt{2}} & 0 \\
\frac{b}{\sqrt{2}} & b & \frac{b}{\sqrt{2}} & \frac{b}{\sqrt{2}} & b & \frac{b}{\sqrt{2}} \\
0 & \frac{a}{\sqrt{2}} & a & -a & \frac{a}{\sqrt{2}} & 0 \\
-\frac{ib}{2a} & -\frac{ia}{2} & iab & -iab & \frac{ia}{2} & \frac{ib}{2a} \\
\frac{ib}{\sqrt{2}} & -ib & \frac{ib}{\sqrt{2}} & -\frac{ib}{\sqrt{2}} & ib & -\frac{ib}{\sqrt{2}} \\
-\frac{b}{2a} & \frac{a}{2} & ab & ab & \frac{a}{2} & -\frac{b}{2a}
\end{array}\right|, \qquad u_2 = \left|\begin{array}{cc}
-\frac{1}{2} & -\frac{1}{2} \\
\frac{i}{2} & -\frac{i}{2}
\end{array}\right|,$$

$$u_3 = \begin{vmatrix} \frac{ib}{\sqrt{2}a} & \frac{ib}{\sqrt{2}a} \\ 0 & 0 \\ \frac{b}{\sqrt{2}a} & -\frac{b}{\sqrt{2}a} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{vmatrix}, \qquad u_4 = \begin{vmatrix} -\frac{ia}{2} & -\frac{ia}{2} \\ 0 & 0 \\ \frac{a}{2} & -\frac{a}{2} \\ -i\sqrt{2}a & i\sqrt{2}a \\ 0 & 0 \\ \sqrt{2}a & \sqrt{2}a \end{vmatrix}, \tag{15}$$

$$u_5 = \begin{vmatrix} \frac{a}{2} & -\frac{a}{2} \\ \frac{i}{2} & -\frac{i}{2} \\ \frac{ia}{2} & \frac{ia}{2} \\ -\frac{a}{\sqrt{2}} & -\frac{a}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} \\ -\frac{ia}{\sqrt{2}} & \frac{ia}{\sqrt{2}} \end{vmatrix}, \qquad u_6 = \begin{vmatrix} -\frac{ia}{2} & \frac{ia}{2} \\ -\frac{1}{2} & -\frac{1}{2} \\ -\frac{a}{2} & -\frac{a}{2} \\ -\frac{ia}{\sqrt{2}} & \frac{ia}{\sqrt{2}} \\ -\frac{i}{2} & -\frac{i}{2} \\ -\frac{a}{\sqrt{2}} & \frac{a}{\sqrt{2}} \end{vmatrix},$$

$$u_7 = \begin{vmatrix} -b & b \\ -ib & -ib \end{vmatrix}, \qquad u_8 = \begin{vmatrix} b & -b \\ -ib & -ib \end{vmatrix}, \qquad u_9 = \begin{vmatrix} ib & ib \\ b & -b \end{vmatrix}, \qquad u_{10} = \begin{vmatrix} -b & -b \\ -ib & ib \end{vmatrix},$$

where $a = \frac{1}{\sqrt{6}}, \quad b = \frac{1}{2\sqrt{2}}$.

The essential point in these transformations consists in changing the metric in the "additional" subspace with spin 1/2, which affects the formulation of the additional conditions and the contribution of the "superfluous" components of the wave function to the equations.

The authors take this opportunity to thank Ya. M. Yaglom for a valuable discussion.

Leningrad State Pedagogical Institute
named after A. I. Herzen

Received
21 XII 1956

REFERENCES

1. I. M. Gel' fand, Ya. M. Yaglom, ZhETF, **18**, 703 (1948).
2. S. N. Gupta, Phys. Rev., **95**, 1334 (1954).

3. M. Petraş, Czechoslov. J. Phys., **5**, 160 (1955).

4. M. Fierz, W. Pauli, Proc. Roy. Soc. London, A **173**, 211 (1939).

5. W. Rarita, J. Schwinger, Phys. Rev., **60**, 61 (1941).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.