Mathematics

K. M. FISHMAN and Yu. N. VALITSKY

1957

SovietRxiv

Abstract

Full Text

Mathematics

K. M. FISHMAN and Yu. N. VALITSKY

ON THE APPLICABILITY OF FREDHOLM' S THEORY TO CERTAIN LINEAR TOPOLOGICAL SPACES

(Presented by Academician V. I. Smirnov on 21 VI 1957)

1. Let $\mathfrak{B}_r$ be a real or complex Banach space depending on a parameter r $(\alpha < r \leq \beta)$. Denote by $\| \|_r$ the norm of $\mathfrak{B}_r$. We shall assume the following concerning these spaces: 1) $\mathfrak{B}_r$ is everywhere a dense linear manifold in $\mathfrak{B}_{r'}$ $(r' < r)$ with respect to the norm $\mathfrak{B}_{r'}$; 2) for every $f \in \mathfrak{B}_r$ one has $\|f\|_{r'} \leq \|f\|_r$ $(\alpha < r' < r \leq \beta)$.

Consider the sets

$$\mathfrak{A}_r = \prod_{r'<r} \mathfrak{B}_{r'} \quad (\alpha < r \leq \beta).$$

By virtue of condition 1), $\mathfrak{A}_r$ is everywhere dense in $\mathfrak{B}_{r'}$ $(r' < r)$. The set $\mathfrak{A}_r$ becomes a linear space if addition and multiplication by numbers are defined in it in the same way as in any $\mathfrak{B}_{r'}$ $(r' < r)$. We define convergence in $\mathfrak{A}_r$ in the following manner: we shall say that $f_n \to 0$ as $n \to \infty$ in $\mathfrak{A}_r$ if, for all $r' < r$, $\|f_n\|_{r'} \to 0$ $(n \to \infty)$. Thus $\mathfrak{A}_r$ becomes a complete linear topological space ([1]). Obviously, for any r and r'', $r < r''$, $\mathfrak{B}_r$ forms a linear manifold in $\mathfrak{A}_r$, and $\mathfrak{A}_{r''}$ in $\mathfrak{B}_r$, while the topology in $\mathfrak{A}_{r''}$ is stronger than the topology in $\mathfrak{A}_r$.

Let $\mathfrak{B}^*_r$ be the space conjugate to $\mathfrak{B}_r$. If $F \in \mathfrak{B}^*_{r'}$ for some $r' < r$, then, by virtue of 2), F is a linear functional also on $\mathfrak{B}_r$, i.e. $\mathfrak{B}^*_{r'} \subset \mathfrak{B}^*_r$ $(r' < r)$, and is there a linear subset. Introduce the linear space

$$\mathfrak{A}^*_r = \sum_{r'<r} \mathfrak{B}^*_{r'},$$

defining in it the same linear operations as in $\mathfrak{B}^*_{r'}$, and call it the space conjugate to $\mathfrak{A}_r$. Every functional $F \in \mathfrak{A}^*_r$ is a linear functional on $\mathfrak{A}_r$. On the other hand, taking into account that the same topology in $\mathfrak{A}_r$ may be obtained by means of a countable system of norms $\| \|_{r'_n}$, $(r > r'_n \to r)$ ([1]), one may assert also the converse, i.e. that every linear (continuous) functional on $\mathfrak{A}_r$ belongs to $\mathfrak{A}^*_r$.

Obviously, for $r < r''$, $\mathfrak{A}_r^* \subset \mathfrak{A}_{r''}^*$, and $\mathfrak{A}_r^* \subset \mathfrak{B}_r^*$; moreover the linear operations in $\mathfrak{A}_r^*$ are the same as in $\mathfrak{A}_{r''}^*$ and in $\mathfrak{B}_r^*$.

$\mathfrak{A}_r^*$ becomes a linear topological space if the notion of convergence is introduced in either of two ways: a) F_n converges (weakly) to zero ($F_n \overset{c}{\to} 0$) if $F_n(f) \to 0$ for every $f \in \mathfrak{A}_r$; b) F_n converges (strongly) to zero ($F_n \Rightarrow 0$) if $F_n(f) \to 0$ uniformly on every bounded set $M \subset \mathfrak{A}_r$ (a set is called bounded if there exist constants $K_{r'} < \infty$ ($r' < r$) such that $\|f\|_{r'} \leq K_{r'}$ for all $f \in M$).

2. Let A be a linear operator mapping $\mathfrak{B}_r$ into $\mathfrak{B}_r$ ($\alpha < r \leq \beta$), and suppose that its action does not depend on r. The operator A induces in each $\mathfrak{A}_r$ a certain linear continuous operator whose action likewise does not depend on r. Denote by A^* the operator conjugate to A, acting in $\mathfrak{B}_r^*$. Its action also does not depend on r. This permits one to consider the operator A^* on the spaces

$$\mathfrak{A}_r^* = \sum_{r'<r} \mathfrak{B}_{r'}^*.$$

The distributivity

of this operator and the independence of its action from r are obvious. The operator A^* is continuous in both topologies in $\mathfrak{A}_r^*$.

3. Let now Fredholm theory be applicable to the operator A in each space $\mathfrak{B}_r$ ([2]). We shall show that in this case all Fredholm theorems also hold in the spaces $\mathfrak{A}_r$ ($\alpha < r \leqslant \beta$).

Theorem 1. *The number of linearly independent solutions of the equations $Af = 0$ and $A^*F = 0$, respectively, in $\mathfrak{A}_r$ and in $\mathfrak{A}_r^*$ is finite and the same for both equations.*

Proof. Let the equation $Af = 0$ have k ($k \leqslant \infty$) linearly independent (l.i.) solutions in $\mathfrak{A}_r$. Then in $\mathfrak{B}_{r'}$, ($r' < r$), the equation $Af = 0$ has $k_1(r') \geqslant k$, $k_1(r') < \infty$, l.i. solutions, and hence $k < \infty$. The equation $A^*F = 0$ in $\mathfrak{B}_{r'}^*$ also has $k_1(r')$ l.i. solutions; therefore in $\mathfrak{A}_r^*$ it has $k_2 \geqslant k_1(r')$ l.i. solutions. Since $\mathfrak{A}_r^* \subset \mathfrak{B}_r^*$, the equation $A^*F = 0$ in $\mathfrak{B}_r^*$ has $k_3 \geqslant k_2$ l.i. solutions, $k_3 < \infty$, and therefore $k_2 < \infty$. Then the equation $Af = 0$ in $\mathfrak{B}_r$ likewise has k_3 l.i. solutions, with $k_3 \leqslant k_1$ by virtue of the inclusion $\mathfrak{B}_r \subset \mathfrak{A}_r$. Hence $k_2 = k (= k_1(r')$ for $r' < r$, which indicates the independence of k and k_2 from r).

Theorem 2. *In order that the equation $Af = g$ $[A^*F = G]$ have a solution in $\mathfrak{A}_r$ ($\mathfrak{A}_r^*$), it is necessary and sufficient that, for every $\Phi \in \mathfrak{A}_r^*$ ($\varphi \in \mathfrak{A}_r$) satisfying the equation $A^*\Phi = 0$ ($A\varphi = 0$), the equality $\Phi(g) = 0$ ($G(\varphi) = 0$) hold.*

Necessity. Let $Af = g$ be solvable in $\mathfrak{A}_r$ and, consequently, in every $\mathfrak{B}_{r'}$, ($r' < r$). If Φ is an arbitrary solution of the equation $A^*\Phi = 0$, $\Phi \in \mathfrak{A}_r^*$, then $\Phi \in \mathfrak{B}_{r-\varepsilon}^*$ ($\varepsilon \leqslant \varepsilon_0$). By virtue of the normal solvability of the operator A in $\mathfrak{B}_{r-\varepsilon}$, we obtain $\Phi(g) = 0$.

Now let the equation $A^*F = G$ be solvable in $\mathfrak{A}_r^*$, and let $\varphi \in \mathfrak{A}_r$ be an arbitrary solution of the equation $A\varphi = 0$. Then for some $\varepsilon > 0$, F and $G \in \mathfrak{B}_{r-\varepsilon}^*$, $\varphi \in \mathfrak{B}_{r-\varepsilon}$, and, by virtue of the normal solvability of the operator A in $\mathfrak{B}_{r-\varepsilon}$, $G(\varphi) = 0$.

Sufficiency. Let $\{\Phi_k\}_1^m$ ($m < \infty$ and independent of r, by Theorem 1) be a maximal l.i. system of solutions of the equation $A^*\Phi = 0$ in $\mathfrak{A}_r^*$ and $\Phi_k(g) = 0$ ($k = 1, \dots, m$), $g \in \mathfrak{A}_r$. For all δ, $\delta \leqslant \delta_0$, $\Phi_k \in \mathfrak{B}_{r-\delta}^*$, $g \in \mathfrak{B}_{r-\delta}$. The system $\{\Phi_k\}_1^m$ is a complete l.i. system of solutions of the equation $A^*\Phi = 0$ in $\mathfrak{B}_{r-\delta}^*$. Thus, the equation $Af = g$ is solvable in every $\mathfrak{B}_{r-\delta}$ ($\delta \leqslant \delta_0$). Then, taking into account the independence of the dimension $m(< \infty)$ of the null subspace of the operator A from r, we obtain its solvability in $\mathfrak{A}_r$.

Let the functional $G \in \mathfrak{A}_r^*$ satisfy the conditions $G(\varphi_k) = 0$ ($k = 1, 2, \dots, m$), where $\{\varphi_k\}_1^m$ is a complete l.i. system of solutions of the equation $A\varphi = 0$ in $\mathfrak{A}_r$. Then for $\delta \leqslant \delta_0$, $G \in \mathfrak{B}_{r-\delta}^*$, $\varphi_k \in \mathfrak{B}_{r-\delta}$. The equation $A\varphi = 0$ has exactly m l.i. solutions in every $\mathfrak{B}_{r-\delta}$; consequently, the equation $A^*F = G$ is solvable in $\mathfrak{B}_{r-\delta}^*$ and, thereby, in $\mathfrak{A}_r^*$.

Theorem 3. *In order that the equation $Af = g$ ($A^*F = G$) have, in $\mathfrak{A}_r$ ($\mathfrak{A}_r^*$), a solution for any g (for any G), it is necessary and sufficient that the homogeneous equation $A\varphi = 0$ ($A^*\Phi = 0$) have only the trivial solution. If these conditions are satisfied, the inverse operator A^{-1} is continuous in $\mathfrak{A}_r$ (the operator $(A^*)^{-1}$ is weakly and strongly continuous in $\mathfrak{A}_r^*$).*

The first assertion follows immediately from Theorems 1 and 2, if one takes into account that the equality $G(\varphi) = 0$ ($\varphi \in \mathfrak{A}_r$) for every $G \in \mathfrak{A}_r^*$ implies $\varphi = 0$.

We shall prove the continuity of the operators A^{-1} and $(A^*)^{-1}$ under the condition of their existence.

The range $R(A)$ of the operator A in any $\mathfrak{B}_{r'}$ ($r' < r$) contains $\mathfrak{A}_r$ and therefore is everywhere dense in $\mathfrak{B}_{r'}$; by virtue of the normal solvability of the operator A in $\mathfrak{B}_{r'}$, $R(A)$ is closed ([1]), and consequently $R(A) = \mathfrak{B}_{r'}$. Thus A^{-1} is continuous in each $\mathfrak{B}_{r'}$ ($r' < r$); since, moreover, it maps $\mathfrak{A}_r$ into $\mathfrak{A}_r$, it is continuous also in the topology of $\mathfrak{A}_r$.

Suppose that $(A^*)^{-1}$ is not a weakly continuous operator in $\mathfrak{A}_r^*$. Then there exists a sequence $\{F_n\} \subset \mathfrak{A}_r^*$ such that $A^*F_n \xrightarrow{\text{sl}} 0$, but F_n does not tend weakly to zero, i.e. for some $f \in \mathfrak{A}_r$, $F_n(f)$ does not tend to zero. Since A is invertible in any $\mathfrak{B}_{r'}$, $r' < r$, we have $A^{-1}f \in \mathfrak{A}_r$, and, by assumption, $A^*F(A^{-1}f) \to 0$, i.e. $F_n(f) \to 0$, which contradicts our supposition. The weak continuity of $(A^*)^{-1}$ is proved. By analogous arguments one proves the strong continuity of $(A^*)^{-1}$.

Consider the operator $A_\lambda = E - \lambda B$, where B is a linear operator on any space $\mathfrak{B}_r$, whose action does not depend on r, $\alpha < r \leq \beta$. Let $\mathcal{G}$ be a domain of the complex (λ)-plane in which A_λ is representable as the sum of a finite-dimensional and an invertible operator in any $\mathfrak{B}_r$, and, in addition, suppose that there exists a point $\lambda_0 \in \mathcal{G}$ at which the operator A_λ is invertible in $\mathfrak{B}_r$. Then the following theorem holds:

Theorem 4. *The number of values λ belonging to $\mathcal{G}$ at which the operator A_λ is not invertible in $\mathfrak{A}_r$ is finite in every closed part of $\mathcal{G}$; these values are the same for all r, $\alpha < r \le \beta$.*

Proof. Let λ be a point of noninvertibility for A_λ; according to Theorem 3, there exists $f \in \mathfrak{A}_r$ such that $A_\lambda f = 0$; since $f \in \mathfrak{B}_{r'}$, $r' < r$, λ is a singular point of the operator A_λ in $\mathfrak{B}_{r'}$. The set of all such values λ for the operator A_λ in $\mathfrak{B}_{r'}$ is discrete in every connected component of the set M_A ([2]), and hence, all the more, in $\mathcal{G} \subset M_A$.

If λ is a point of noninvertibility of A_λ in $\mathfrak{A}_r$, then the same is true in the spaces $\mathfrak{A}_{r'}$, $\alpha < r' \le \beta$.

Example. Let

$$E_r(z) = \sum_0^\infty \alpha_n^{-1}(r) z^n \qquad (0 < \alpha_n < \infty;\ \alpha < r \le \beta)$$

be an analytic function in the disk C_{r^p} $(|z| < r^p)$; let $\mathfrak{B}_r$ be the Banach space $Z^p_{E_n}$, i.e. the collection of all functions

$$f(z) = \sum_n a_n z^n,$$

for which

$$\|f\|_r = \left(\sum_n |a_n|^p \alpha_n(r) \right)^{1/p} < \infty,$$

where $p \ge 1$. Suppose that $\alpha_n(r)$ increases monotonically with increasing r. It is obvious that the spaces $\mathfrak{B}_r$ satisfy conditions 1), 2) of item 1. To each functional $F \in \mathfrak{B}_r^*$ there corresponds the function

$$\sum_{n=0}^\infty \frac{b_n}{\zeta^{n+1}},$$

where

$$\sum_0^\infty |b_n|^q \alpha_n^{1-q}(r) < \infty \quad (p > 1)$$

or

$$\sup_n |b_n| \alpha_n^{-1}(r) < \infty \quad (p = 1)$$

(here

$$\frac{1}{p} + \frac{1}{q} = 1$$

). In this case

$$F\left(\sum_n a_n z^n\right) = \sum_n a_n b_n.$$

The space $\mathfrak{A}_r$ in our case is the space of all functions analytic in the disk C_r, and the topology coincides with the one usually adopted for these spaces. The space $\mathfrak{A}_r^*$ is isomorphic to the space of all functions analytic in the domain $|z| \geq r$ ([4]).

Consider an operator B, acting in any $\mathfrak{B}_r$ and given by the matrix $[\gamma_{mn}]_{m,n=0}^{\infty}$, on which we impose the following requirement:

$$\gamma_{mn} = \gamma'_{mn}(r) + \gamma''_{mn}(r) \quad (\alpha < r \leq \beta),$$

where:

1) for any finite sequence of complex numbers $M_1, M_2, \dots, M_k$ the inequality holds

$$\sum_{n=0}^{\infty} \left| \sum_{m=0}^{k} \gamma''_{mn} M_m \right|^p \alpha_n(r) \leq [\theta(r)]^p \sum_{n=0}^{k} |M_n|^p \alpha_n(r), \qquad \theta(r) < 1, \quad \alpha < r \leq \beta;$$

2) there exist two matrices $[c_n^{(i)}(r)]_{\substack{n=0,1,\dots \\ i=1,2,\dots,s<\infty}}$ and $[d_n^{(i)}(r)]_{\substack{n=0,1,\dots \\ i=1,2,\dots,s<\infty}}$,

$$\sum_{n=0}^{\infty} |c_n^{(i)}(r)|^p \alpha_n(r) < \infty \qquad (i = 1, 2, \dots, s),$$

$$\sum_{n=0}^{\infty} |d_n^{(i)}|^q \alpha_n^{1-q}(r) < \infty \qquad (i = 1, 2, \dots, s), \quad \text{if } p > 1$$

(in the case $p = 1$, the latter condition must be replaced by the following: $\sup |d_n^{(i)}(r)| \alpha_n^{-1}(r) < \infty$), $\alpha < r \leqslant \beta$, so that the relation

$$\gamma'_{mn} = \sum_{i=1}^{s} d_m^{(i)}(r) c_n^{(i)}(r).$$

holds.

Define the action of the operator B in $\mathfrak{B}_r$ as follows:

$$B\left(\sum_n a_n z^n\right) = \sum_n \left(\sum_m \gamma_{mn} a_m\right) z^n.$$

The operator B is an integral operator with kernel

$$\sum_{m,n=0}^{\infty} \frac{\gamma_{nm} z^m}{\zeta^{n+1}}.$$

To the operator B^* there corresponds the matrix $[\gamma_{nm}]$, transposed to $[\gamma_{mn}]$. By virtue of our assumptions, the operator B is decomposed into the sum of operators $B' + B''$, to which correspond the matrices $[\gamma'_{mn}]$ and $[\gamma''_{mn}]$, $\|B''\| < 1$, and B' is a finite-dimensional operator. Therefore the operator

$$A = E - B = (E - B'') - B'$$

is the sum of an invertible and a finite-dimensional operator. By Theorems 1-3, for the operator A acting in the analytic spaces $\mathfrak{A}_r$, Fredholm theory holds.

If one discards the condition $\theta(r) < 1$ and considers, instead of A, the operator

$$A_\lambda = E - \lambda B = (E - \lambda B'') - \lambda B',$$

taking into account that for $|\lambda| < \left[\sup_{\alpha < r \leqslant \beta} \theta(r)\right]^{-1}$ the conditions of Theorem 4 are satisfied, then the assertion of this theorem is applicable to the indicated domain.

The conditions set out here for A_λ are satisfied, in particular, if we assume $p = 1$, $\alpha_n(r) = r^n$, and

$$\overline{\lim}_{n\to\infty} \sum_{m=0}^{\infty} \gamma_{nm} r^{m-n} \leqslant \theta(r).$$

In this case, Fredholm theory for the operator A_λ in the analytic spaces $\mathfrak{A}_r$ was constructed by M. A. Evgrafov by another method [3].

It is not hard to find other examples fitting the proposed scheme.

Chernivtsi State University

Received
20 VI 1957

References

[1] I. M. Gel' fand, *Uspekhi Mat. Nauk*, **11**, no. 6 (1956).
[2] S. M. Nikol' skii, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **7**, no. 3 (1943).
[3] M. A. Evgrafov, *Trudy Moskov. Mat. Obshch.*, **5** (1956).
[4] A. I. Markushevich, *Matem. Sbornik*, **17** (59), no. 2 (1945).

Note: Figure translations are in progress. See original paper for figures.