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Abstract

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INVESTIGATION OF THE STABILITY PROBLEM FOR SYSTEMS OF EQUATIONS WITH HOMOGENEOUS RIGHT-HAND SIDES

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In the present paper conditions are derived for the asymptotic stability of the zero solution of a system of ordinary differential equations with homogeneous right-hand sides, and exact estimates are given for the distance of an integral curve from the equilibrium position; a number of applications of the results obtained is also set forth.

Definition 1. A real single-valued continuous function $X(x_1, \dots, x_n)$, defined in E_n, will be called **homogeneous of order** μ and denoted by $X^{(\mu)}$, $\mu = p/q$, where p and q are natural numbers, q odd, if for any real constant c the equality

$$X(cx_1, \dots, cx_n) = c^{\mu} X(x_1, \dots, x_n)$$

holds; here the quantity c^{μ} will be regarded as positive when p is even, and when p is odd—as real, having the sign of c.

Definition 2. A real single-valued continuous function $V(x_1, \dots, x_n)$, defined in E_n, will be called **positive-homogeneous of order** $m > 0$ and denoted by $V^{[m]}$, if for any real quantity c the equality

$$V(cx_1, \dots, cx_n) = |c|^m V(x_1, \dots, x_n)$$

holds; here the quantity $|c|^m$ is regarded as positive.

Consider the system of ordinary differential equations

$$\frac{dx_s}{dt} = X_s^{(\mu)}(x_1, \dots, x_n) \qquad (s = 1, \dots, n). \tag{1}$$

Further, by

$$X = X(t, X^{(0)}) \tag{2}$$

we shall denote an integral curve of system (1) such that $X(0, X^{(0)}) = X^{(0)}$, where X is a real n-dimensional vector $(x_1, \dots, x_n)$. It is clear that, together with the integral curve (2), system (1) has a family of integral curves, depending on one arbitrary real constant c, representable in the form

$$X = cX(c^{\mu-1}t, X^{(0)}) = X(t, cX^{(0)}).$$

Theorem 1. 1) *For p even, the zero solution of system (1) cannot be asymptotically stable.*

2) *If p is odd and $\mu \neq 1$, then the zero solution of system (1) can be asymptotically stable only under real perturbations.*

3) *If $\mu = 1$, then the zero solution of system (1) can be asymptotically stable under arbitrary complex perturbations.* With the aid of the results contained in work (1), one can show that the integral curves of system (1) satisfy the inequality

$$\left|X(t, X^{(0)})\right| < At^{-\alpha} \tag{3}$$

for $\left|X^{(0)}\right| = 1$, where A and α are positive constants, if only

the zero solution of system (1) is asymptotically stable and $X_s^{(\mu)} \in C_\nu$, i.e., the functions $X_s^{(\mu)}$ are continuously differentiable with respect to all their arguments in E_n up to order $\nu \geqslant 1$, inclusive.

Thus, in what follows, inequality (3) for $X_s^{(\mu)} \in C_\nu,\ \nu \geqslant 1$, should be regarded as equivalent to asymptotic stability.

Theorem 2. If the integral curves of system (1) satisfy inequality (3), then there exist a constant $m > \mu - 1$ and two functions $W^{[m]}$ and $V^{[m+1-\mu]}$ possessing the following properties:

1) the functions $V^{[m-\mu+1]}$ and $-W^{[m]}$ are positive definite;

2) the function $V^{[m-\mu+1]}(X(t, X^{(0)}))$ is continuously differentiable with respect to t, and the equality

$$\frac{dV^{[m-\mu+1]}}{dt} = W^{[m]}.$$

holds.

If, in addition, $X_s^{(\mu)} \in C_\nu,\ \nu \geqslant 1$, then the function $W^{[m]}$ can be chosen so that the function $V^{[m-\mu+1]} \in C_\nu$ is the unique solution of the system

$$\sum_{i=1}^{n} \frac{\partial V}{\partial x_i} X_i^{(\mu)} = W^{[m]}; \qquad \sum_{i=1}^{n} x_i \frac{\partial V}{\partial x_i} = (m+1-\mu)V, \tag{4}$$

determined by the condition $V(0, \dots, 0) = 0$.

Theorem 3. In order that inequality (3) hold, it is necessary and sufficient that the following inequalities be satisfied for the solution of system (1):

1) for $\mu = 1$,

$$p_1|X^{(0)}|e^{-p_2 t} \leqslant |X(t, X^{(0)})| \leqslant q_1|X^{(0)}|e^{-q_2 t};$$

2) for $\mu > 1$ and $\mu \neq 2k/q$,

$$\frac{c_1|X^{(0)}|}{\sqrt[\mu-1]{1 + c_2 t\,|X^{(0)}|^{\mu-1}}} \leqslant |X(t, X^{(0)})| \leqslant \frac{d_1|X^{(0)}|}{\sqrt[\mu-1]{1 + d_2 t\,|X^{(0)}|^{\mu-1}}};$$

3) for $\mu \in (0,1)$, every solution of system (1), if (3) is assumed, after a finite interval of time reaches the origin. Until that moment the inequalities

$$\sqrt[1-\mu]{a_1|X^{(0)}|^{1-\mu} - a_2 t} \leqslant |X(t, X^{(0)})| \leqslant \sqrt[1-\mu]{b_1|X^{(0)}|^{1-\mu} - b_2 t}.$$

hold.

In the indicated inequalities $a_i,\ b_i,\ p_i,\ q_i,\ c_i,\ d_i$ $(i = 1, 2)$ are positive constants.

Remark 1. The possibility of case 3) was pointed out in [2].

Remark 2. The estimates given in Theorem 3, in a certain sense, cannot be improved, i.e., the method of computing the constants $a_i,\ b_i,\ c_i,\ d_i,\ p_i,\ q_i$ $(i = 1, 2)$ is such that examples can be given in which these inequalities become equalities.

Consider the system

$$\frac{dx_s}{dt} = X_s^{(\mu)} + f_s(x_1, \dots, x_n, t) \quad (s = 1, \dots, n). \tag{5}$$

We shall assume that system (5) has a solution $X = X(t, X^{(0)}, t_0)$ for every $X^{(0)} \in E_n$ and $t_0 \in (-\infty, +\infty)$.

Theorem 4. Let the functions $X_s^{(\mu)} \in C_\nu$. In order that all solutions of system (5), for any choice of functions f_s, $|f_s| \leqslant c|X|^\lambda$ for $|X| \geqslant R$, $\lambda \leqslant \mu$, where R is a sufficiently large positive quantity, $c > 0$ is sufficiently small when $\lambda = \mu$, be bounded for $t \geqslant t_0$, it is necessary and sufficient that (3) hold for every solution of system (1).

Theorem 5. If $X_s^{(\mu)} \in C$, and (3) holds, then the zero solution of system (5) will be asymptotically stable for any functions f_s such that

$$|f_s(X, t)| \leq c_1|X|^\lambda$$

for $|X| \leq h$, where $\lambda \geq \mu$, $h > 0$ is sufficiently small, and $c_1 > 0$ is sufficiently small when $\lambda = \mu$. Moreover, any solution of system (5) beginning for sufficiently small $|X^{(0)}|$, for all $t \geq t_0$, satisfies estimates of the same form as in Theorem 3, if in them t is replaced by $t - t_0$, and $X(t, X^{(0)})$ by $X(t, X^{(0)}, t_0)$.

The study of system (1) makes it possible to advance the solution of the stability question in a number of doubtful cases.

Consider the system

$$\frac{dx_s}{dt} = X_s(x_1, \dots, x_k, y_1, \dots, y_n) \quad (s = 1, \dots, k);$$

$$\frac{dy_j}{dt} = \sum_{i=1}^{n} p_{ji} y_i + Y_j(x_1, \dots, x_k, y_1, \dots, y_n) =$$

$$= \Phi_j(x_1, \dots, x_k, y_1, \dots, y_n) \quad (j = 1, \dots, n), \tag{6}$$

where the functions X_s, Y_j are holomorphic in a neighborhood of

$$x_1 = \dots = x_k = y_1 = \dots = y_n = 0$$

and their expansions contain no terms linear with respect to x_s, y_j.

Suppose that the roots of the equation $|P - \lambda E| = 0$, $\{P\}_{ik} = p_{ik}$, have negative real parts. Denote by

$$u_j(x_1, \dots, x_k)$$

the analytic functions satisfying the system of equations

$$\Phi_j(x_1, \dots, x_k, y_1, \dots, y_n) = 0 \quad (j = 1, \dots, n).$$

Theorem 6. If

$$X_s(x_1, \dots, x_k, u_1, \dots, u_n) \equiv 0 \quad (s = 1, \dots, k),$$

then system (6) has k holomorphic integrals and the zero solution of this system is stable in the sense of Lyapunov.

Denote by

$$\overline{X}_s^{(\mu)}(x_1, \dots, x_k)$$

the forms of lowest degree in the expansion of the functions

$$X_s(x_1, \dots, x_k, u_1, \dots, u_n)$$

in powers of the quantities $x_1, \dots, x_k$.

Theorem 7. If the zero solution of the system

$$\frac{dx_s}{dt} = \overline{X}_s^{(\mu)} \quad (s = 1, \dots, k)$$

is asymptotically stable, then the zero solution of system (6) is also asymptotically stable, and any solution beginning in a sufficiently small neighborhood of the point

$$x_1 = \dots = x_k = y_1 = \dots = y_n = 0$$

satisfies the inequalities

$$p_1 |Z^{(0)}| e^{-p_2 t} \leq |Z(t, Z^{(0)})| \leq \frac{c_1 |Z^{(0)}|}{\sqrt[\mu-1]{1 + c_2 t |Z^{(0)}|^{\mu-1}}} \quad \text{for } t > 0;$$

where $c_i, p_i > 0$ $(i = 1, 2)$ are constants;

$$Z(t, Z^{(0)}) = (x_1, \dots, x_k, y_1, \dots, y_n)$$

is a solution of system (6);

$$Z^{(0)} = (x_1^{(0)}, \dots, x_k^{(0)}, y_1^{(0)}, \dots, y_n^{(0)}).$$

We note that Theorem 6 is established with the aid of the results contained in § 65 of work ([3]).

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REFERENCES

1. N. N. **Krasovskii**, Prikl. matem. i mekh., **19**, no. 5 (1955).
2. V. I. **Zubov**, Prikl. matem. i mekh., **17**, no. 4 (1953).
3. A. M. **Lyapunov**, Complete collected works, **2**, Publishing House of the Academy of Sciences of the USSR, 1956.

Note: Figure translations are in progress. See original paper for figures.