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Abstract

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O. M. BELOTSERKOVSKII

FLOW PAST A CIRCULAR CYLINDER WITH A DETACHED SHOCK WAVE

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The problem of flow past bodies with a detached shock wave has been studied only in a very approximate formulation ([3–5]), etc. The use of electronic computing machines makes it possible to obtain definitive results with the required degree of accuracy for the problem in an exact formulation.

1. Formulation of the problem. Let a plane-parallel supersonic flow ($M_\infty > 1$) of an ideal gas impinge with constant velocity w_∞ on a circular cylinder. In front of the cylinder a shock wave arises, whose shape and position are not known in advance. We introduce dimensionless quantities, referring the velocity to the maximum velocity; the pressure and density to the stagnation pressure and density before the shock wave; linear dimensions to the radius of the cylinder, and consider the system of equations of motion, continuity, and energy:

$$\operatorname{rot}\mathbf{w}\times\mathbf{w}+\nabla\frac{w^2}{2}+\frac{\nabla(kp)}{\rho}=0,\qquad \nabla(\rho\mathbf{w})=0,\qquad \mathbf{w}\nabla\frac{p}{\rho^{\varkappa}}=0,\tag{1}$$

where w, p, ρ are the velocity, pressure, and density behind the shock wave; $\varkappa$ is the adiabatic exponent; $k = (\varkappa - 1)/2\varkappa$ (for air $\varkappa = 1.40$ and $k = 1/7$).

Introducing polar coordinates (r, ϑ) (Fig. 1), and replacing one of the equations of motion by the Bernoulli integral, we obtain a system equivalent to (1); together with the equation for the stream function (ψ), the new system will have the form:

$$\frac{\partial rH}{\partial r}+\frac{\partial S}{\partial\vartheta}-g=0,\qquad \frac{\partial rh}{\partial r}+\frac{\partial t}{\partial\vartheta}=0,$$

$$\varphi=\frac{p}{\rho^{\varkappa}}=\varphi(\psi),\qquad \frac{d\psi}{d\vartheta}=\rho\left(v\frac{dr}{d\vartheta}-ru\right),\tag{2}$$

where

$$H=kp+\rho u^2,\qquad S=\rho uv,\qquad g=kp+\rho v^2,\qquad h=\tau u,$$

Fig. 1

Figure 1: Fig. 1

$$t = \tau v, \qquad \tau = (1-w^2)^{\frac{1}{\varkappa-1}}, \qquad p = (1-w^2)\rho, \qquad \rho = \tau \varphi^{-\frac{1}{\varkappa-1}}, \tag{3}$$

u, v are the components of the velocity w along r and ϑ. In this system the unknowns are the functions u, v, φ, ψ.

The boundary conditions on the body ($r = 1$) have the form:

$$u(\vartheta) = 0, \qquad \psi(\vartheta) = 0, \qquad \varphi(\vartheta) = \varphi(0) = \text{const}. \tag{4}$$

On the shock wave, where $r(\vartheta) = 1+\varepsilon(\vartheta)$, the known relations for the tangential and normal components of velocity, pressure, density, and the stream function, together with the obvious geometrical relation along the wave, can be written in the form

$$u(\vartheta) = F_2 \sin\vartheta - F_1 \cos\vartheta, \qquad v(\vartheta) = F_1 \sin\vartheta + F_2 \cos\vartheta,$$

$$p(\vartheta) = \frac{4\varkappa}{\varkappa^2-1}(1-w_\infty^2)^{\frac{\varkappa}{\varkappa-1}} \left[\frac{w_\infty^2 \cos^2\chi}{1-w_\infty^2} - \frac{(\varkappa-1)^2}{4\varkappa}\right], \tag{5}$$

$$\varphi(\vartheta) = \frac{4\varkappa}{\varkappa^2-1} \left[\left(\frac{\varkappa-1}{\varkappa+1}\right) \frac{1-w_\infty^2 \sin^2\chi}{w_\infty^2\cos^2\chi}\right]^{\varkappa} \left[\frac{w_\infty^2 \cos^2\chi}{1-w_\infty^2} - \frac{(\varkappa-1)^2}{4\varkappa}\right],$$

$$\psi(\vartheta) = w_\infty (1-w_\infty^2)^{\frac{1}{\varkappa-1}} (1+\varepsilon)\sin\vartheta, \qquad \frac{d\varepsilon(\vartheta)}{d\vartheta} = (1+\varepsilon)\tan(\vartheta-\chi),$$

where $\varepsilon(\vartheta)$ is the distance from the surface of the body to the wave along the ray $\vartheta = \text{const}$; χ is the angle between the tangent to the wave and the vertical (Fig. 1); $\psi(0) = 0$; F_1 and F_2 are known functions of χ, M_∞.

Fig. 1

2. Method of solution. To solve the problem posed, the method of integral relations (1) was applied; it consists of the following. Between the wave and the body we draw $N-1$ lines equidistant along r,

$$\xi = \xi_i = \text{const} \qquad (\xi = (r-1)/\varepsilon(\vartheta), \quad 0 < \xi < 1),$$

which divide the region of integration into N strips. We denote all quantities on the body, where $\xi = \xi_0 = 0$, by the subscript 0; on the i-th line, where

$$\xi = \xi_i = [N-(i-1)]/N,$$

by the subscript i; and on the wave ($\xi = \xi_1 = 1$) by the subscript 1. Integrating the first two equations of system (2) along an arbitrary ray $\vartheta = \text{const}$, from the surface of the body to the boundary of each of the strips, we then obtain $2N$ independent integral relations:

$$r_i H_i - H_0 + \frac{d}{d\vartheta} \int_1^{1+\xi_i \varepsilon(\vartheta)} S(r, \vartheta)\, dr - S_i \xi_i \frac{d\varepsilon(\vartheta)}{d\vartheta} - \int_1^{1+\xi_i \varepsilon(\vartheta)} g(r, \vartheta)\, dr = 0; \quad (6)$$

$$r_i h_i - h_0 + \frac{d}{d\vartheta} \int_1^{1+\xi_i \varepsilon(\vartheta)} t(r, \vartheta)\, dr - t_i \xi_i \frac{d\varepsilon(\vartheta)}{d\vartheta} = 0; \quad (7)$$

$$i = 1, 2, \dots, N; \qquad r_i(\vartheta) = 1 + \xi_i \varepsilon(\vartheta); \qquad H_i = H(r_i, \vartheta); \qquad H_0 = H(1, \vartheta); \qquad S_i = S(r_i, \vartheta);$$

$$h_i = h(r_i, \vartheta); \qquad h_0 = h(1, \vartheta); \qquad t_i = t(r_i, \vartheta).$$

We now approximate any integrand function $f(r, \vartheta)$ by interpolation polynomials of degree N in r, taking the strip boundaries as interpolation nodes,

$$f(r, \vartheta) = \sum_{m=0}^{N} a_m(\vartheta) \left[\frac{r-1}{\varepsilon(\vartheta)} \right]^m, \quad (8)$$

where $a_m(\vartheta)$ will depend linearly on the values of the corresponding function at the strip boundaries.

Substitute (8) into the integral relations (6), (7) and carry out the integration. Then, writing the last two equations of system (2) along each of the $N-1$ lines $\xi = \xi_j,\ j = 2, 3, \dots, N$ (on the wave and on the body, φ and ψ are found from the boundary conditions), and taking into account the last equation of (5), we obtain an approximating system consisting of $4N-1$ equations for the unknowns $v_0, u_j, v_j, \varphi_j, \psi_j, \varepsilon, \chi$. Thus the problem is reduced to a system of $3N$ ordinary differential equations and $N-1$ finite relations.

From the condition of symmetry we obtain the following boundary conditions: for $\vartheta = 0$,

$$v(r, 0) = \psi(r, 0) = \chi(0) = 0.$$

However, the system obtained in the neighborhood–

of sound lines will have singular points. The requirement that the motion be continuous at these points also gives us the missing conditions for the determinacy of the problem. These conditions are similar to those obtained by S. A. Khristianovich[2] from an exact system of partial differential equations for

Fig. 2

Figure 2: Fig. 2

certain cases of vortical flows. The solution constructed in this way will satisfy all the conditions of the given problem, including the boundary conditions on the body and on the wave.

The solution of the approximating system was carried out numerically for different numbers of intermediate lines: in the first approximation ($N = 1$) no intermediate line was introduced, and S, g, t were approximated linearly from their values on the body and on the wave. The three unknown functions v_0, ε, χ were found from three differential equations. In the second approximation ($N = 2$) one intermediate line was introduced; in the third ($N = 3$), two, and so on. Agreement of the results with the required accuracy in the last two approximations indicates practical convergence of the calculation (Fig. 3).

Fig. 2

We shall explain the calculation technique using the example of the second approximation. The approximations have the form:

$$f(r,\vartheta) = f_0 + [4f_2 - f_1 - 3f_0]\,\frac{r-1}{\varepsilon(\vartheta)} + 2\,[-2f_2 + f_1 + f_0]\left[\frac{r-1}{\varepsilon(\vartheta)}\right]^2,$$

where f_1, f_2, f_0 are the values of $f(r,\vartheta)$, respectively, on the wave, on the middle line, and on the body. The approximating system is as follows:

$$\frac{d\varepsilon}{d\vartheta} = (1+\varepsilon)\,\mathrm{tg}(\vartheta - \chi), \qquad \frac{d\chi}{d\vartheta} = \Phi^{(2)}, \qquad \frac{dv_0}{d\vartheta} = \frac{E_0^{(2)}}{1/6 - v_0^2},$$

$$\frac{du_2}{d\vartheta} = \frac{1}{-\dfrac{1}{\tau_2 v_2 \varphi_2^{\chi-1}}}\left(F_3 - F_4 u_2 \varphi_2^{-\frac{1}{\chi-1}} + \frac{5}{2}F_5\frac{d\psi_2}{d\vartheta}\right),$$

$$\frac{dv_2}{d\vartheta} = \frac{E_2^{(2)}}{\dfrac{1+5u_2^2}{6} - \omega_2^2}, \qquad \frac{d\psi_2}{d\vartheta} = \frac{\rho_2}{2}\left[v_2\frac{d\varepsilon}{d\vartheta} - (2+\varepsilon)u_2\right], \qquad \varphi_2 = \varphi_1(\psi_1)\big|_{\psi_1=\psi_2},$$

where $\Phi^{(2)}, E_0^{(2)}, E_2^{(2)}, F_3, F_4, F_5$ are known functions of $\vartheta, u_i, v_i, \varphi_i, \varphi_i$; $i = 0, 1, 2$; $\chi = 1.40$.

The integration of the system is carried out from $\vartheta = 0$, where $v_0 = v_2 = \chi = \psi_2 = 0$, while $\varepsilon_0 = \varepsilon(0)$ and $u_2(0)$ are two unknown parameters. At the points

Fig. 3

Figure 3: Fig. 3

where $v_0^2 = 1/6$ and $\omega_2^2 = (1+5u_2^2)/6$, the derivatives $dv_0/d\vartheta$ and $dv_2/d\vartheta$ become infinite, and from the condition of continuity of the motion at these points it must be that $E_0^{(2)} = 0$ and $E_2^{(2)} = 0$. In each subsequent approximation one parameter $u_i(0)$ and one condition are added: for $\omega_i^2 = (1 + 5u_i^2)/6$, $E_i = 0$. Thus the points where the line $\omega_i^2 - (1 + 5u_i^2)/6 = 0$ intersects the boundaries of the strip or the surface of the body will be singular points of our system (singularities of the "saddle" type).

In conclusion, let us find on the wave the angle δ_k between the streamline ($\psi =$ const) and the line $\omega = \omega_k =$ const. Writing, in rectangular coordinates, the equations of continuity, vorticity, and two expressions for the total derivatives along,

wave from the velocity components, we obtain a system of equations for determining 4 partial derivatives of these components. Then from the condition $dw = 0$ we determine the required angle δ_k, which in plane flow, for a given w_k, depends, it turns out, only on M_∞—thus the shape of the body does not affect this angle (in the axisymmetric case the angle δ_k also depends on the curvature of the wave). For the sonic line ($w_k = 1/\sqrt{6}$), the change of the angle δ with M_∞ is given in Table 1.

Table 1

M_∞	1.00	1.125	1.25	1.50	1.69	2.00	2.50	3.00	4.00	5.00	6.00	7.00	8.00
δ°	90.000	96.332	96.894	95.915	90.000	73.606	46.174	30.697	18.914	14.804	12.888	11.838	11.189

An expression has also been obtained for δ_k on the body and in the field, but here this angle is determined through w, dw/ds (s is the length of the arc along $\psi =$ const), the radius of curvature of the streamline, and the value of the vorticity at the point under consideration.

Fig. 3. I—first approximation; II—second approximation; III—third approximation; a—experiment

3. Results of the calculations. Calculations of the flow past a circular cylinder at various M_∞ were carried out on the BESM electronic machine according to the I, II, and partly the III approximations.

In Fig. 1, for the case $M_\infty = 3.0$, the shock wave, the sonic line ($M = 1$), and the characteristics are plotted according to the III approximation. The flow pattern obtained according to the II approximation is very close to the one plotted.

In Fig. 2, waves and sonic lines are plotted for the cases $M_\infty = 3.0$ (III approximation), 4.0 and 5.0 (II approximation). The angle of inclination of the sonic line at the point of its intersection with the wave agrees well with the data of Table 1.

Fig. 3 illustrates the convergence of the method by approximations for $M_\infty = 3.0$; here a comparison is also given with the results of an experiment carried out by G. M. Ryabinkov (the ratio of the pressure on the body to the pressure at the critical point is denoted by $p_0(\vartheta)/p_0(0)$).

Analysis of the calculations shows that already the II approximation gives sufficiently accurate results. The calculation data make it possible to construct the pattern of pressure on the surface and outside the body, the sonic line, the position and shape of the shock wave, etc. By an analogous procedure, the problem of flow with a detached shock wave past plane or spatial bodies of arbitrary shape possessing an axis of symmetry is solved.

Computing Center
Academy of Sciences of the USSR

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.