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**Abstract**

**Full Text**

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**SOME UNSTEADY PLANE AND SPATIAL GAS FLOWS**

*(Presented by Academician N. N. Bogolyubov, 12 IX 1956)*

We shall consider some types of plane and spatial unsteady isentropic gas flows. First we shall consider plane gas flows.

The basic equations of these flows have the form:

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + c^2 \frac{\partial \ln \rho}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + c^2 \frac{\partial \ln \rho}{\partial y} &= 0, \\ \frac{\partial \ln \rho}{\partial t} + u \frac{\partial \ln \rho}{\partial x} + v \frac{\partial \ln \rho}{\partial y} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned} \quad (1)$$

(the notation is standard).

Considering potential flows, when

$$u = \frac{\partial \varphi}{\partial x}, \quad v = \frac{\partial \varphi}{\partial y}, \quad (2)$$

and proceeding from the first two equations of system (1), we arrive at the Bernoulli equation

$$\frac{\partial \varphi}{\partial t} + \frac{q^2}{2} + i = 0, \quad (3)$$

where  $q = \sqrt{u^2 + v^2}$ ,  $i = \int \frac{dp}{\rho} = \int c^2 d \ln \rho$ .

Hence it is easy to determine  $\rho = \rho(\varphi)$ , which will make it possible to represent the third equation of system (1) in the form:

$$\frac{\partial}{\partial t} \left( \frac{\partial \varphi}{\partial t} + \frac{q^2}{2} \right) + u \frac{\partial}{\partial x} \left( \frac{\partial \varphi}{\partial t} + \frac{q^2}{2} \right) + v \frac{\partial}{\partial y} \left( \frac{\partial \varphi}{\partial t} + \frac{q^2}{2} \right) = c^2 \Delta \varphi, \quad (4)$$

where, in the case of the law  $p = A\rho^k$ ,

$$c^2 = (k-1)i = -(k-1) \left( \frac{\partial \varphi}{\partial t} + \frac{q^2}{2} \right).$$

However, it is useless to seek a solution of this equation.

Let us consider a class of self-similar motions, putting

$$\frac{x}{t} = z_1; \quad \frac{y}{t} = z_2. \quad (5)$$

Such flows may occur in various problems of gas outflow or in the flow around certain surfaces. In this case the equations of system (1) take the form:

$$\begin{aligned} \frac{\partial u}{\partial z_1}(u - z_1) + \frac{\partial u}{\partial z_2}(v - z_2) + c^2 \frac{\partial \ln \rho}{\partial z_1} &= 0, \\ \frac{\partial v}{\partial z_1}(u - z_1) + \frac{\partial v}{\partial z_2}(v - z_2) + c^2 \frac{\partial \ln \rho}{\partial z_2} &= 0, \\ \frac{\partial \ln \rho}{\partial z_1}(u - z_1) + \frac{\partial \ln \rho}{\partial z_2}(v - z_2) + \frac{\partial u}{\partial z_1} + \frac{\partial v}{\partial z_2} &= 0. \end{aligned} \quad (6)$$

Putting  $\varphi = t\psi$ , we write (3) in the form:

$$t \frac{\partial \psi}{\partial t} + \psi + \frac{q^2}{2} + i = 0 \quad \text{or} \quad \frac{\partial \psi}{\partial z_1} z_1 + \frac{\partial \psi}{\partial z_2} z_2 = \psi + \frac{q^2}{2} + i. \quad (7)$$

The conditions of potentiality of the flow (2) take the form

$$u = \frac{\partial \psi}{\partial z_1}, \quad v = \frac{\partial \psi}{\partial z_2}. \quad (8)$$

Therefore (7) can be written in the form

$$uz_1 + vz_2 = \psi + \frac{q^2}{2} + i. \quad (9)$$

We write the last equation of the system (6) in the form

$$\frac{\partial i}{\partial z_1}(u - z_1) + \frac{\partial i}{\partial z_2}(v - z_2) + c^2 \left( \frac{\partial u}{\partial z_1} + \frac{\partial v}{\partial z_2} \right) = 0 \quad (10)$$

or, since

$$di = d(uz_1 + vz_2) - (d\psi + u du + v dv)$$

and taking (8) into account, we obtain

$$di = du(z_1 - u) + dv(z_2 - v). \quad (11)$$

We now represent (10) in the form:

$$\frac{\partial u}{\partial z_1} [(u - z_1)^2 - c^2] + \left( \frac{\partial v}{\partial z_1} + \frac{\partial u}{\partial z_2} \right) (u - z_1)(v - z_2) + \frac{\partial v}{\partial z_2} [(v - z_2)^2 - c^2] = 0 \quad (12)$$

(where it must be borne in mind that  $\frac{\partial v}{\partial z_1} = \frac{\partial u}{\partial z_2}$ ).

We shall now interchange the dependent and independent variables, namely, we shall regard  $z_1, z_2$  as functions of  $u, v$ ; then (12) takes the form:

$$\frac{\partial z_2}{\partial v} [(u - z_1)^2 - c^2] - \left( \frac{\partial z_2}{\partial u} + \frac{\partial z_1}{\partial v} \right) (u - z_1)(v - z_2) + \frac{\partial z_1}{\partial u} [(v - z_2)^2 - c^2] = 0. \quad (13)$$

The potentiality conditions (8) are written in the form

$$z_1 = \frac{\partial \psi}{\partial u}, \quad z_2 = \frac{\partial \psi}{\partial v}, \quad (14)$$

where  $\psi = \psi(u, v)$ .

Equation (10) now becomes  $di = d\psi - \frac{dq^2}{2}$ , whence

$$i + \frac{q^2}{2} = \psi. \quad (15)$$

Thus, (13) is reduced to the equation

$$\frac{\partial^2 \psi}{\partial u^2} \left[ \left( v - \frac{\partial \psi}{\partial v} \right)^2 - c^2 \right] - 2 \frac{\partial^2 \psi}{\partial u \partial v} \left( u - \frac{\partial \psi}{\partial u} \right) \left( v - \frac{\partial \psi}{\partial v} \right) + \frac{\partial^2 \psi}{\partial v^2} \left[ \left( u - \frac{\partial \psi}{\partial u} \right)^2 - c^2 \right] = 0, \quad (16)$$

where for the law  $p = A\rho^k$

$$c^2 = (k - 1) \left[ \psi - \frac{q^2}{2} \right].$$

We have arrived at a quasilinear equation of the second order with respect to the function  $\psi$ . Let us write the equations of the characteristics in the  $u, v$  plane (in the hodograph plane) for this equation:

$$\frac{dv}{du} = \frac{-(u - z_1)(v - z_2) \pm c\sqrt{(u - z_1)^2 + (v - z_2)^2 - c^2}}{(v - z_2)^2 - c^2}. \quad (17)$$

For  $c = 0$  the characteristics pass into the equation of the trajectory or, for the variables  $z_1, z_2$ , into the streamlines; the equation of these lines in the  $(u, v)$  plane is

$$\frac{dv}{du} = -\frac{u - z_1}{v - z_2}. \quad (18)$$

This equation is easily obtained from (11), putting in it  $i = 0$ ,  $di = 0$ .

The general equation of the trajectories has the form

$$dt = \frac{dx}{u} = \frac{dy}{v}. \quad (19)$$

In the case under consideration (5)

$$dt = \frac{z_1 dt + t dz_1}{u} = \frac{z_2 dt + t dz_2}{v}.$$

whence it follows that

$$\frac{dz_1}{u - z_1} = \frac{dz_2}{v - z_2} = d \ln t.$$

The equation

$$\frac{dz_1}{u - z_1} = \frac{dz_2}{v - z_2} \quad (20)$$

is also the equation of a streamline in the plane  $(z_1, z_2)$  (this equation is not identical with equation (18)).

Finding the general solution of (16) is impossible, although this equation is simpler than equation (4). We shall try to find some particular solution of the basic system of equations for self-similar motions.

Let us first consider spatial self-similar gas flows, and then, as a special case, plane ones. Let  $u, v, w, c$  depend on

$$z_1 = \frac{x}{t}, \quad z_2 = \frac{y}{t}, \quad z_3 = \frac{z}{t}. \quad (21)$$

Then in the independent variables  $z_1, z_2, c$  we arrive at the equations:

$$\begin{aligned}
& (u - z_1) \left[ \frac{\partial u}{\partial z_1} \frac{\partial z_2}{\partial c} - \frac{\partial u}{\partial c} \frac{\partial z_2}{\partial z_1} \right] + (v - z_2) \frac{\partial u}{\partial c} + (w - z_3) \left[ \frac{\partial z_2}{\partial c} \frac{\partial u}{\partial z_3} - \frac{\partial z_2}{\partial z_3} \frac{\partial u}{\partial c} \right] - c^2 \frac{d \ln \rho}{dc} \frac{\partial z_2}{\partial z_1} = 0, \\
& (u - z_1) \left[ \frac{\partial v}{\partial z_1} \frac{\partial z_2}{\partial c} - \frac{\partial v}{\partial c} \frac{\partial z_2}{\partial z_1} \right] + (v - z_2) \frac{\partial v}{\partial c} + (w - z_3) \left[ \frac{\partial z_2}{\partial c} \frac{\partial v}{\partial z_3} - \frac{\partial z_2}{\partial z_3} \frac{\partial v}{\partial c} \right] + c^2 \frac{d \ln \rho}{dc} = 0, \\
& (u - z_1) \left[ \frac{\partial w}{\partial z_1} \frac{\partial z_2}{\partial c} - \frac{\partial w}{\partial c} \frac{\partial z_2}{\partial z_1} \right] + (v - z_2) \frac{\partial w}{\partial c} + (w - z_3) \left[ \frac{\partial z_2}{\partial c} \frac{\partial w}{\partial z_3} - \frac{\partial z_2}{\partial z_3} \frac{\partial w}{\partial c} \right] - c^2 \frac{d \ln \rho}{dc} \frac{\partial z_2}{\partial z_3} = 0, \\
& - (u - z_1) \frac{\partial z_2}{\partial z_1} + (v - z_2) - (w - z_3) \frac{\partial z_2}{\partial z_3} \\
& + \frac{dc}{d \ln \rho} \left[ \frac{\partial u}{\partial z_1} \frac{\partial z_2}{\partial c} - \frac{\partial u}{\partial c} \frac{\partial z_2}{\partial z_1} + \frac{\partial v}{\partial c} + \frac{\partial z_2}{\partial c} \frac{\partial w}{\partial z_3} - \frac{\partial z_2}{\partial z_3} \frac{\partial w}{\partial c} \right] = 0.
\end{aligned} \tag{22}$$

Let  $u = u(c)$ ,  $v = v(c)$ ,  $w = w(c)$ ; assuming that

$$z_2 = z_1 f_1(c) + z_3 f_3(c) + f_2(c), \tag{23}$$

we arrive at the system of equations:

$$F + f_1 c^2 \frac{d \ln \rho}{du} = 0, \quad F - c^2 \frac{d \ln \rho}{dv} = 0, \quad F + f_3 c^2 \frac{d \ln \rho}{dw} = 0, \tag{24}$$

$$F + f_1 \frac{du}{d \ln \rho} - \frac{dv}{d \ln \rho} + f_3 \frac{dw}{d \ln \rho} = 0,$$

where  $F = u f_1 + w f_3 + f_2 - v$ . After simple transformations one easily obtains the equations:

$$du = \frac{f_1 c d \ln \rho}{\sqrt{1 + f_1^2 + f_3^2}}, \tag{25}$$

$$dv = -\frac{c^2 d \ln \rho}{\sqrt{1 + f_1^2 + f_3^2}}, \tag{26}$$

$$dw = \frac{f_3 c d \ln \rho}{\sqrt{1 + f_1^2 + f_3^2}}, \tag{27}$$

$$f_2 = v - (u f_1 + w f_3) \pm c \sqrt{1 + f_1^2 + f_3^2}. \tag{28}$$

From equations (25), (26), (27) one can obtain the relation

$$(du)^2 + (dv)^2 + (dw)^2 = (c d \ln \rho)^2, \tag{29}$$

which is a relation along the characteristics and in the present case is satisfied throughout the entire domain of the solution found; this solution may be called special.

Thus, we have arrived at a solution with two arbitrary functions  $f_1(c)$ ,  $f_3(c)$ , prescribing on any two surfaces

$$z_2 = \bar{z}_2(z_1, z_3), \quad u = \bar{u}(c), \quad v = \bar{v}(c), \quad w = \bar{w}(c). \quad (30)$$

These functions can be determined. If  $f_2 \equiv 0$ , then

$$y = xf_1(c) + zf_3(c), \quad (31)$$

and we arrive at Busemann's generalized solution for stationary flow past ruled or conical surfaces. Consequently, nonstationary flows with  $f_3 \equiv 0$  do not exist.

If plane flows are considered, it is necessary to set  $f_3(c) \equiv 0$ ; then

$$f_2 = v - uf_1 \pm c\sqrt{1 + f_1^2}. \quad (32)$$

Prescribing on some line

$$z_2 = \bar{z}_2(z_1), \quad u = \bar{u}(c), \quad v = \bar{v}(c), \quad (33)$$

we easily determine the arbitrary function

$$f_1(c) = \frac{(\bar{z}_1 - \bar{u})(\bar{z}_2 - \bar{v}) \pm c\sqrt{(\bar{z}_1 - \bar{u})^2 + (\bar{z}_2 - \bar{v})^2 - c^2}}{(\bar{z}_1 - \bar{u})^2 - c^2}. \quad (34)$$

Let us consider the important special case when

$$\bar{z}_1 - \bar{u} \pm c = 0; \quad (35)$$

then

$$z_2 - v = \pm cf_1 + c\sqrt{1 + f_1^2}, \quad (36)$$

whence

$$f_1(c) = \frac{c^2 - (\bar{z}_2 - \bar{v})^2}{2c(\bar{z}_2 - \bar{v})}, \quad (37)$$

$$z_2 - v = \frac{[c^2 - (\bar{z}_2 - \bar{v})^2](z_1 - \bar{u}) \pm c[c^2 + (\bar{z}_2 - \bar{v})^2]}{2c(\bar{z}_2 - \bar{v})}. \quad (38)$$

Next, we determine, under the conditions  $c = a_1$ ;  $u = a_2$ ;  $v = a_3$ ,

$$u = \pm \int \frac{c^2 - (\bar{z}_2 - \bar{v})^2}{c^2 + (\bar{z}_2 - \bar{v})^2} c d \ln \rho, \quad v = \mp \int \frac{2c(\bar{z}_2 - \bar{v})}{c^2 + (\bar{z}_2 - \bar{v})^2} c d \ln \rho. \quad (39)$$

Equations (38) and (39) give the solution of the indicated special case.

If  $f_2 \equiv 0$ , then we arrive at Prandtl-Meyer's generalized solution for the flow of a stationary gas stream past some profile or angle (1). Consequently, nonstationary flows with  $f_2 \equiv 0$  do not exist.

Generalizations of the indicated solutions (spatial and plane) can easily be found if one assumes that

$$u = t^{a-1}\xi, \quad v = t^{a-1}\eta, \quad w = t^{a-1}\theta, \quad c = t^{a-1}\omega, \quad (40)$$

where  $\xi, \eta, \theta, \omega$  depend on  $z_1, z_2, z_3$  given by (21).

The indicated solutions may have application, in particular, in the study of the process of dispersal of the detonation products of shaped charges.

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## References

1. L. Landau, E. Lifshitz, *Mechanics of Continuous Media*, § 107, 1953.

*Note: Figure translations are in progress. See original paper for figures.*

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