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Abstract

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MATHEMATICAL PHYSICS

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ON THE SOLUTION OF THE STEFAN PROBLEM BY REDUCTION TO A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

(Presented by Academician S. L. Sobolev on 27 IV 1957)

The Stefan problem is formulated as a problem of matching two temperature fields (1) and (2) in the presence of a special boundary condition (3) on the moving interface:

$$\frac{\partial u_1(x,t)}{\partial t} = a_1^2 \frac{\partial^2 u_1(x,t)}{\partial x^2}, \qquad 0 < x < \xi(t); \tag{1}$$

$$\frac{\partial u_2(x,t)}{\partial t} = a_2^2 \frac{\partial^2 u_2(x,t)}{\partial x^2}, \qquad \xi(t) < x < l; \tag{2}$$

$$\alpha_1 \left.\frac{\partial u}{\partial x}\right|_{x=\xi} - \alpha_2 \left.\frac{\partial u_2}{\partial x}\right|_{x=\xi} = \frac{d\xi}{dt}; \tag{3}$$

$$u_1(0,t) = \Phi_1(t); \quad u_2(0,t) = \Phi_2(t); \quad u_1[\xi(t),t] = u_2[\xi(t),t] = T_0 = 0;$$

$$u_1(x,0) = \varphi_1(x)\ (0 < x < \xi_0); \qquad u_2(x,0) = \varphi_2(x)\ (\xi_0 < x < l); \qquad \xi_0 \neq 0.$$

The main difficulty in solving the Stefan problem consists in the fact that, because of condition (3), it belongs to the class of nonlinear problems.

Alongside numerous methods using simplifications of the problem ([1–4,8]), a method has been proposed which leads to obtaining an exact solution ([5–7]). However, the author does not attempt to carry it through to a numerical result.

§ 1. **Reduction of the Stefan problem to an infinite system of ordinary differential equations.** For convenience of proof we shall consider one zone $(0 < x < \xi)$, taking into account the influence of the other in condition (3) temporarily by means of a certain bounded function $q = q(t)$. We assume that $\varphi(x)$ and $\Phi(t)$ have continuous first derivatives and that $\varphi(0) = \Phi(0)$; $\varphi''(x)$ is integrable. Let

$$u(x,t) = V(x,t) + \Phi(t)\frac{\xi - x}{\xi}; \qquad V(0,t) = V(l,t) = 0. \tag{4}$$

Suppose (as will be proved below) that $V(x,t)$ can be expanded in a Fourier series on the interval $(0,\xi)$ for fixed t:

$$V(x,t) = \frac{2}{\xi}\sum_{i=1}^{\infty} A_i(t)\sin\frac{i\pi x}{\xi}. \tag{5}$$

Let us form for A_k and ξ an infinite system of differential equations (multiplying (1) by $\sin(i\pi x/\xi)$ and integrating from 0 to ξ):

$$A_k' = -\frac{a^2k^2\pi^2}{\xi^2}A_k - k\xi'\left[\frac{2}{\xi}\sum_{i=1}^{\infty} A_i W_{ik} - \frac{\Phi(t)(-1)^k}{k^2\pi}\right] - \frac{\Phi'(t)\xi}{k\pi}, \tag{6}$$

$$\xi' = \frac{2a\pi}{\xi^2}\sum_{i=1}^{\infty} iA_i(-1)^i - \frac{a\Phi(t)}{\xi} - q, \tag{7}$$

where $W_{ik} = (-1)^{i+k+1}\dfrac{i}{i^2-k^2}$, $i \neq k$; $\quad W_{ik} = -\dfrac{1}{4k}$, $i = k$.

Let us show that the solution of the system (6) and (7) is obtained by passage to the limit as $n \to \infty$ from the solution of the system consisting of $n+1$ equations

$$A_k^{(n)\prime} = -\frac{a^2k^2\pi^2}{\xi^{(n)2}}A_k^{(n)} - k\xi^{(n)\prime}\left[\frac{2}{\xi^{(n)}}\sum_{i=1}^{n} A_{iW_{ik}} - \frac{\Phi(t)(-1)^k}{k^2\pi}\right] - \frac{\Phi'(t)\xi^{(n)}}{k\pi}; \tag{8}$$

$$\xi^{(n)\prime} = \frac{2a\pi}{\xi^{(n)2}}\sum_{i=1}^{n} iA_i^{(n)}(-1)^i - \frac{a\Phi(t)}{\xi^{(n)}} - q. \tag{9}$$

The existence of a solution of the system (8), (9) follows from the boundedness of the partial derivatives of the right-hand sides with respect to all the unknown functions.

§ 2. **Estimate of the solution of (8) and (9).** Making in (8) the substitution $A_k^{(n)} = \dfrac{1}{k^3}C_k$ and solving it as a first-order differential equation, we obtain*

$$|C_k| \leq M + \frac{\max|r_k|}{\min p_k}\left(1 - e^{-\min p_k \cdot t}\right), \tag{10}$$

where

$$p_k = \frac{a^2k^2\pi^2}{\xi^{(n)2}}; \qquad r_k = k^4\xi^{(n)\prime}\left[\frac{2}{\xi^{(n)}}\sum_{i=1}^{n}\frac{C_{iW_{ik}}}{i^3} - \frac{\Phi(t)(-1)^k}{k^2\pi}\right] - \frac{\Phi'(t)\xi^{(n)}}{\pi}k^2.$$

Take some natural number m^{**} and introduce the notation:

$$\max_{k\le m}|C_k| = N; \qquad \max_{k>m}|C_k| = F; \qquad \max\left|\xi^{(n)} + \frac{1}{\xi^{(n)}}\right| = G > 1***.$$

All constants occurring in the inequalities, their combinations, and also bounded expressions will be denoted by the same letter M. We estimate the sums in (10) (for $1 \le i \le m$ and $m+1 \le i \le n$):

$$N \le MG^6\left(N + \frac{F}{m} + 1\right)^2 m^4 t; \qquad \frac{F}{m} \le MG^6\left(N + \frac{F}{m} + 1\right)^2 \frac{1}{m},$$

$$\xi^{(n)} + \frac{1}{\xi^{(n)}} \le \frac{1}{\sqrt[3]{-M(N+F/m+1)t + C}}.$$

Thus one can obtain:

$$z \le K + \beta\frac{z^2}{(1-\omega z)^2},$$

where $z = N + \frac{F}{m} + 1$; $K = M + \frac{M}{m} + 1$; $\beta = M\left(m^4 t + \frac{1}{m}\right)$; $\omega = Mt$. Taking sufficiently large m and sufficiently small t, one can ensure that β and ω are sufficiently small. Hence it follows that, for any $\varepsilon > 0$, there is a time interval T such that, for $0 \le t \le T$, $z \le K + \varepsilon$, i.e. $N \le M$, $F \le M$. In what follows, t will be considered only in this interval. From this estimate it follows that M does not depend on n. Thus, $G \le M$.

§ 3. **Passage to the limit to an infinite system of differential equations.** Consider the system (8), (9) and an analogous system obtained by replacing n by $n+p$ (p an arbitrary natural number). We introduce the notation

$$C_k^{(n+p)} = \widetilde{C}_k; \quad \xi^{(n+p)} = \tilde{\xi}; \quad \widetilde{C}_k - C_k = \Delta C_k; \quad \tilde{\xi}_k - \xi_k^{(n)} = \Delta\xi \quad (k \le n).$$

Using for ΔC_k formula (10), integrating the equation for $\Delta\xi$, and assuming that for $k \le m$, $\max|\Delta C_k| \le H$; for

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Fig. 1

Figure 1: Fig. 1

* By $\min p$ is meant the least value of $p(s)$ for $0 \le s \le t$. All other extrema appearing are also considered for the indicated time interval.

** The introduction of m is due to the fact that, in the course of the proof, it is not possible to estimate all C_k separately. However, the final estimate is unified for all C_k.

*** Powers of $\xi^{(n)} + 1/\xi^{(n)}$ can majorize both positive and negative powers of $\xi^{(n)}$.

for $k > m$, $\max |\Delta C_k| = L$, one can obtain

$$H + \frac{L}{m} \le \gamma \left(H + \frac{L}{m} \right) + \frac{\gamma}{n},$$

where

$$\gamma = M \left(t e^{Mt} + 1 \right) \left(\frac{1}{m} + m^2 t \right).$$

Taking m sufficiently large and t sufficiently small, one can ensure that $\gamma < 1$. Thus,

$$H + \frac{L}{m} \le \frac{M}{n}.$$

Since $H > 0$ and $L > 0$, it follows that

$$|\Delta C_k| \le \frac{M}{n}, \qquad |\Delta \xi| \le \frac{M}{n}.$$

Passing to the limit in equations (8) and (9) as $n \to \infty$, one can verify that the sequences $A_k^{(n)}$ and $\xi^{(n)}$ converge uniformly. Thus a solution of the system (6) and (7) has been obtained. From the estimates for $A_k^{(n)}$ and $\xi^{(n)}$ it follows that $V(x, t)$, expressed by formula (5), is continuous in x and t and has a continuous derivative with respect to x. Therefore the resulting $u(x, t)$ satisfies the condition at the phase-separation boundary (3). To establish (1), it remains to prove the existence and continuity of $\partial u / \partial t$ and $\partial^2 u / \partial x^2$ (or $\partial V / \partial t$ and $\partial^2 V / \partial x^2$) on the interval $(0, \xi)$.

Fig. 1

Let us write (6) in the form

$$C_k' = -p k^2 D_k - k^2 \left[R_1 (-1)^k - R_2 \right] + R_3,$$

where $p(t)$, $R_1(t)$, and $R_2(t)$ do not depend on k, and

$$|R_3(t)| \le M k \ln k.$$

Make the substitution

$$C_k = D_k - \frac{R_1(-1)^k - R_2}{p}.$$

Integrating, we obtain

$$|D_k| \le S(t)\frac{\ln k}{k}; \qquad |D'_k| \le Mk\ln k,$$

where $S(t)$ does not depend on k. Substituting C_k in (5) and in (4), and taking into account the estimates for D_k and D'_k, we obtain that $u(x,t)$ is continuously differentiable twice with respect to x and once with respect to t.

§ 4. **Uniqueness of the solution.** Suppose that there exists a second solution $\overline{u}(x,t)$ of equation (1), satisfying the same boundary conditions and having continuous second derivative with respect to x and first derivative with respect to t. Perform on it the same transformations as on $u(x,t)$ in the preceding paragraphs. From the continuity of $\overline{u}_x(x,t)$ it follows that

$$\overline{A}_k \le \frac{M}{k^3}$$

and

$$\overline{C}_k \le M.$$

As before,

$$\left|\frac{1}{\overline{\overline{\xi}}} + \frac{1}{\xi}\right| \le M.$$

Denote $\overline{C}_k - C_k$ by δC_k, and $\overline{\xi} - \xi$ by $\delta\xi$. Since C_i and $\overline{C}_i$, and consequently also δC_i, are bounded, δC_i has an upper bound. Assuming that, for $k \le m$, $|\sup \delta C_k| \le H$, while for $k > m$, $|\sup \delta C_k| \le L$, using (10) for δC_k and integrating the equation for $\delta\xi$, we obtain

$$\overline{H} + \frac{\overline{L}}{m} \le \gamma\left(\overline{H} + \frac{\overline{L}}{m}\right),$$

where $\gamma < 1$. Hence $\delta C_k \equiv 0$, and also $\delta\xi \equiv 0$.

§ 5. **The final system of differential equations.** Considering both zones together

$$\left(u_2(x,t) = \Phi_2(t)\frac{x-\xi}{l-\xi} + \frac{2}{l-\xi}\sum_{i=1}^{\infty} B_i(t)\sin\frac{i\pi(x-\xi)}{l-\xi}, \quad \xi < x < l\right),$$

we obtain the complete system of differential equations

$$A'_k = -\frac{a_1^2k^2}{\eta^2}A_k - k\eta'\left[\frac{2}{\eta}\sum_{i=1}^{n} A_{iW_{ik}} - \frac{\Phi_1(t)(-1)^k}{k^2}\right] - \frac{\Phi'_1(t)\eta}{k};$$

Figure 2: Temperature field at $t = 2957$ h; vertical axis H (m).

Figure 2: Figure 2: Temperature field at $t = 2957$ h; vertical axis H (m).

$$B'_k = -\frac{a_2^2 k^2}{(L-\eta)^2} B_k - k\eta' \left[\frac{2}{L-\eta} \sum_{i=1}^{\infty} B_{iU_{ik}} - \frac{\Phi_2(t)}{k^2} \right] + \frac{\Phi'_2(t)(L-\eta)}{k} (-1)^k;$$

$$\eta' = \beta_1 \left[\frac{2}{\eta} \sum_{i=1}^{\infty} iA_i(-1)^i - \Phi_1(t) \right] \frac{1}{\eta} - \beta_2 \left[\frac{2}{L-\eta} \sum_{i=1}^{n} iB_i + \Phi_2(t) \right] \frac{1}{L-\eta},$$

where

$$\beta_1 = \frac{\alpha_1}{\pi^2}; \quad \beta_2 = \frac{\alpha_2}{\pi^2}; \quad \eta = \frac{\xi}{\pi}; \quad L = \frac{l}{\pi};$$

$$U_{ik} = \frac{i}{i^2 - k^2}, \quad i \neq k; \qquad U_{kk} = \frac{1}{4k}.$$

Fig. 2

The results of solving the system for $n = 1, 2, 3, 4$ in the case when

$$u_1(0,t) = -3–7.5 \sin \frac{2\pi}{T} t, \quad u_2(l,t) = u_2(19.4;t) = -3, \quad \xi_0 = 0.41 \text{ m}, \quad T = 8760 \text{ h};$$

$$\alpha_1 = 0.29 \cdot 10^{-4} \text{ m}^2/\text{h} \cdot \text{deg}; \quad \alpha_2 = 0.45 \cdot 10^{-4} \text{ m}^2/\text{h} \cdot \text{deg}; \quad a_1^2 = 2.14 \cdot 10^{-3} \text{ m}^2/\text{h};$$

$$a_2^2 = 1.38 \cdot 10^{-3} \text{ m}^2/\text{h},$$

are presented in Figs. 1 and 2.

It follows from the graphs that, beginning with $n = 2$, the convergence of the sequence of approximate solutions is practically sufficiently rapid. For practical computations one may restrict oneself to the case $n = 2$. This case can quite easily be computed on hand-operated calculating machines.

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CITED LITERATURE

1. Lame, Clapeyron, Stefan, *Ann. de Chim. et de Phys.* (1831).
2. Stefan, *Monatsschr. f. Mathem. u. Phys.* (1890).
3. L. S. Leibenzon, *Handbook of Petroleum Mechanics*, 1931.
4. L. S. Leibenzon, *Izv. AN SSSR, Ser. Geogr. i Geofiz.*, No. 6 (1939).
5. L. I. Rubinshtein, *Izv. AN SSSR, Ser. Geogr. i Geofiz.*, 11, No. 1 (1947).
6. L. I. Rubinshtein, *DAN*, 58, No. 2 (1947).
7. L. I. Rubinshtein, *Izv. AN SSSR, Ser. Geogr. i Geofiz.*, 11, No. 6 (1947).
8. A. N. Tikhonov, A. A. Samarskii, *Equations of Mathematical Physics*, 1947.

Note: Figure translations are in progress. See original paper for figures.