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1957

Abstract

Full Text

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HOMOLOGIES OF SPACES OF CLOSED CURVES

(Presented by Academician P. S. Aleksandrov, 17 VI 1957)

In this note we study spaces of closed curves on Riemannian manifolds and the relation of these spaces to closed geodesics. The homologies of spaces of closed curves on certain manifolds, in particular on spheres,* are computed.

Let M be a closed Riemannian manifold. From the closed curves lying in the manifold M, one can form a number of spaces.

An element of the space $L = L_M$ of closed rectifiable curves with a marked point will be taken to be a mapping $f(t)$ of the interval $[0; 1]$ into the manifold M, satisfying the condition $f(0) = f(1)$ and defining a rectifiable curve on which the parameter t is proportional to arc length (t is the reduced length). In the space L a metric is introduced by the formula

$$r(f, g) = \sup_{0 \le t \le 1} \rho(f(t), g(t)) + |J(f) - J(g)|$$

(where ρ denotes distance in the manifold M, and $J(f)$ the length of the curve f). In the space L acts the group K of real numbers reduced modulo 1. Namely, to a number $\alpha \in K$ and a curve $f \in L$ there is put in correspondence the curve $T_\alpha f$, defined by the mapping $T_\alpha f(t) = f(t - \alpha)$ (the number $t - \alpha$ being reduced modulo 1).

The space $P = P_M$ of oriented closed rectifiable curves of the manifold M is defined as the space obtained from the space L by identifying elements of L equivalent with respect to the group K.

In the spaces L and P an involution $\mathfrak{P}$ is defined, which assigns to each curve the same curve traversed in the opposite direction (i.e. to the curve given by the mapping $f(t)$ there is assigned the curve given by the mapping $f(1-t)$). Identifying in the spaces L and P points equivalent with respect to this involution, we obtain the space $\overline{L} = \overline{L}_M$ of nonoriented closed rectifiable curves with a marked point and the space $\overline{P} = \overline{P}_M$ of nonoriented closed rectifiable curves.

The homotopy type of the spaces $L, P, \overline{L}, \overline{P}$ depends only on the topological structure of the manifold M (more precisely, only on the homotopy type of the manifold M).

In the spaces $L, P, \overline{L}, \overline{P}$ there are distinguished subsets consisting of one-point curves (curves of length 0). We shall denote these subsets respectively by $T, N, \overline{T}, \overline{N}$.

* The homologies of spaces of closed curves were computed earlier by Morse ([1]), Fet ([2]), and Bott ([3]). However, Morse incorrectly computed the type numbers of a nondegenerate closed geodesic (his result is incorrect for multiple geodesics), as a consequence of which the Betti numbers he computed for the sphere cannot be regarded as correct. This makes Fet's results unfounded. Bott's computation of the oriented circular Betti numbers of the sphere is likewise erroneous.

Theorem 1. The circular connectivities of the manifold M (circular connectivities ([1])) are equal to the Betti numbers* mod 2 of the space $\bar{P}$ modulo the subset N. The oriented circular connectivities of the manifold M (sensed circular connectivities ([3])) are equal to the Betti numbers mod 2 of the space P modulo the subset N.

On the spaces $L, P, \bar{L}, \bar{P}$ there is given a continuous functional J—the length of a curve. The critical points of the functional J in the spaces $L, P, \bar{L}, \bar{P}$ are the closed geodesics of the manifold M.

Theorem 2. The type numbers m^k of the critical set corresponding to a nondegenerate closed geodesic of index i in the space L or $\bar{L}$, over an arbitrary field, are given by the formulas $m^i = 1$, $m^{i+1} = 1$ or 0, $m^k = 0$ for $k \neq i, i+1$ (modulo 2 in all cases $m^{i+1} = 1$).

Sketch of proof. A small neighborhood of this critical set can be, in the usual way, deformed, without increasing the lengths of curves, into the set consisting of geodesic polygons with a fixed number of sides. To the function defined on the manifold consisting of polygons, Bott' s results ([3]) on the type numbers of a nondegenerate critical manifold are applicable.

Theorem 3. The type numbers of an s-fold nondegenerate closed geodesic g in the space P or $\bar{P}$, over an arbitrary coefficient field, are determined by the index of this geodesic and by the indices of those geodesics of which it is an iterate. Namely, if by h one denotes a simple closed geodesic whose s-fold iterate is the geodesic g, by $i(d)$ the index of the d-fold iterate of the geodesic h, and by d_p the greatest divisor of the number s not divisible by the prime number p, then, in the case when the number $i(2) - i(1)$ is even, the type numbers of the geodesic g are given by the formulas: $m_0^{i(s)} = 1$, $m_0^k = 0$ for $k \neq i(s)$; $m_p^k = 1$ for $i(d_p) + 1 \leq k \leq i(s)$; $m_p^k = 0$ for the remaining k, and in the case when the number $i(2) - i(1)$ is odd—by the formulas: $m_0^k = m_p^k = 0$ for all k, if $p \neq 2$, $m_2^k = 1$ for $i(d_2) + 1 \leq k \leq i(s)$; $m_2^k = 0$ for the remaining k.

(Here m_0^k, m_p^k denote the k-th type number of the geodesic g over the field of rational numbers and, respectively, over the prime field modulo p.)

The homology groups of the space L are conveniently computed with the aid of the fibration obtained by assigning to each curve of the space L a point marked on it. The base of this fibration is the manifold M, and the fiber is the space Ω of closed paths in the manifold M.

In the case when the manifold M is homeomorphic to the n-dimensional sphere S^n, by means of this fibration one can completely compute the cohomology ring of the space L. Indeed, in this case the structure of the spectral sequence of the indicated fibration is easily clarified, for example, by comparison with the spectral sequence of the fibration of the subspace of the space L consisting of circles with a marked point, into fibers consisting of circles passing through a given point. For an odd-dimensional sphere the spectral sequence of the cohomology rings with coefficients in any ring A turns out to be trivial, and, consequently,

$$H(L, A) = H(S^n, A) \otimes H(\Omega, A).$$

In the case when the sphere S^n is even-dimensional, this result holds modulo 2. For integral

* The homology and cohomology groups of a topological space are defined as the direct (respectively inverse) limit of the spectrum of homology groups (respectively cohomology groups) of the bicompacts contained in this space. The homology algebra (the i-dimensional group) of a space X with coefficients in a ring A is denoted by $H(X, A)$ (respectively $H^i(X, A)$). In the case when the coefficient domain A is the field of rational numbers, we write simply $H(X)$ and $H^i(X)$. The cohomology algebra with bicompact supports and coefficients in the field of rational numbers is denoted by $H_c(X)$.

coefficients, we have $E_2 \approx E_n$, $E_{n+1} \approx E_\infty$, and $d_n x_n^2 e_{2i+1} = 0$, $d_n x_n^2 e_{2i} = 2x_n^2 e_{2i-1}$ (by e_i is denoted a generating element of the group $H^{i(n-1)}(\Omega, Z)$). Hence it follows that $H^i(L, Z) = Z$ for $i = 0$, $i = (2k-1)(n-1)$, and $i = (2k-1)(n-1)+1$; $H^i(L, Z) = Z_2$ for $i = 2k(n-1)+1$, and $H^i(L, Z) = 0$ in the remaining cases (k runs through the natural numbers). Multiplication in the cohomology ring $H(L, Z)$ is trivial.

With the aid of this same fibration one can compute the algebra $H(L_M)$ in the case where the manifold M is simply connected and the algebra $H(M)$ is a tensor product of algebras with one generator. In this case the space M can be mapped in such a way onto the topological product $M_1 \times \cdots \times M_r$ of polyhedra $M_1, \dots, M_r$, for each of which the algebra $H(M_i)$ has a single generator, that the algebra $H(M_1 \times \cdots \times M_r)$ is mapped isomorphically onto $H(M)$. It is not difficult to verify that under the corresponding mapping of the space L_M into the space $L_{M_1 \times \cdots \times M_r}$ the algebra $H(L_{M_1 \times \cdots \times M_r})$ is mapped isomorphically onto the algebra $H(L_M)$. Further, it is evident that the space $L_{M_1 \times \cdots \times M_r}$ is homeomorphic to the space $L_{M_1} \times \cdots \times L_{M_r}$, and consequently, in order to finish the computation of the algebra $H(L_M)$, it is enough to compute the algebra $H(L_{M_i})$. It turns out that in the case where the dimension s of the unique generator of the algebra $H(M_i)$ is odd, the algebra $H(L_{M_i})$ is a free anticommutative algebra with two generators, one of which has dimension s, and the other $s-1$. In the case where the algebra $H(M_i)$ has a generator x of even dimension s, subject to the relation $x^m = 0$, the algebra $H(L_{M_i})$ has generators t, e_j, f_k (j and k run through the natural numbers) of dimensions respectively $s, (j-1)(ms-2)+s-1,\ k(ms-2)+s$, connected by the anticommutativity relations and by the relations $e_{jx}^{m-1} = f_{kx}^{m-1} = x^m = e_{je}k = 0,\ e_{jf}k = xe_{j+k},\ f_{jf}k = xf_{j+k}$.

The cohomology rings $H(\bar{L}, A)$ can, by analogous methods, be computed without difficulty in all cases in which the rings $H(L, A)$ were computed above.

In computing the homologies of the space P it is convenient to use the following lemma.

Lemma. *Let on the space X act the group K of real numbers reduced modulo 1, let X' be a subset of the space X, invariant with respect to the group K, containing all points that remain fixed under each of the transformations of the group K. Denote by Y (respectively Y') the space obtained from X (respectively X') by identifying points equivalent with respect to the group K.*

There exists a spectral sequence with second term

$$E_2 = H(Y \bmod Y') \otimes H(S^1)$$

and limiting term

$$E_\infty = GH(X \bmod X'),$$

and also a spectral sequence with second term

$$E_2 = H(B_K) \otimes H(X \bmod X')$$

and limiting term

$$E_\infty = GH(X \bmod Y').$$

(By B_K is denoted the classifying space of the group K.)

Sketch of proof. Let us first examine the case where X is bicompact and there exists such a natural number n that to every element of the group K not belonging to the subgroup

$$K_n = \left\{0; \frac{1}{n}; \frac{2}{n}; \ldots; \frac{n-1}{n}\right\}$$

there corresponds a transformation having no fixed points in $X \setminus X'$. Consider the spaces $\hat{X}$ and $\hat{X}'$, obtained from the spaces X and X' by identifying points equivalent with respect to the group K_n. On the space $\hat{X} \setminus \hat{X}'$ the group K/K_n acts without fixed points. The spectral sequences of cohomology algebras with bicompact supports over the field of rational numbers obtained-

of the principal fiber space $\hat{X} \setminus \hat{X}'$ with base $Y \setminus Y'$ and group K/K_n, isomorphic to K, and of the fiber space associated with it[5], homotopically equivalent to $Y \setminus Y'$, with base B_K and fiber $\hat{X} \setminus \hat{X}'$, are the desired ones. Indeed, the algebra $\bar{H}_c(\hat{X} \setminus \hat{X}')$ is isomorphic to the algebra $H_c(X \setminus X')$ (see (4)) and, consequently, to the algebra $\bar{H}(X \bmod X')$.

The proof of the lemma in the general case is carried out by means of passage to the limit.

With the aid of the lemma just proved, one can completely compute the cohomology algebras $H(P)$, $H(P \setminus N)$, and $H(P \bmod N)$ in the case when the manifold M is homeomorphic to a sphere, and also in the case when the algebra $H(M)$ is an exterior algebra with an odd number of generators.

We describe here the structure of the algebra $H(P \setminus N)$ in the case when the manifold M is homeomorphic to the n-dimensional sphere. If $n = 2m+1$, then the algebra $H(P \setminus N)$ has two generators e and x, of dimensions respectively $n-1$ and 2, and subject, apart from the anticommutativity relations, only to the relation $ex = 0$. If $n = 2m$, then the algebra $H(P \setminus N)$ has generators $x, f_1, f_2, f_3, \ldots$ of dimensions $2, d, n-1, 3(n-1), 5(n-1), \ldots$, connected by the anticommutativity relations and the relations $f_i f_j = f_k x = 0$.

The cohomology algebras $H(\bar{P})$, $H(\bar{P} \setminus \bar{N})$, $H(\bar{P} \bmod \bar{N})$ are easily computed by using the fact that they are isomorphic to the subalgebras of the algebras $H(P)$, $H(P \setminus N)$, $H(P \bmod N)$, consisting of elements invariant with respect to the involution φ^* (see (4)). If the manifold M is homeomorphic to the sphere S^{2m+1}, then on the generators of the algebra $H(P \setminus N)$ the automorphism φ^* acts as follows: $\varphi^* e = -e$, $\varphi^* x = x$. If the manifold M is homeomorphic to the sphere S^{2m}, then the action of the automorphism φ^* in the algebra $H(P \setminus N)$ is determined by the formulas $\varphi^* x = x$, $\varphi^* f_i = (-1)^i f_i$.

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Received
15 VII 1957

CITED LITERATURE

[1] M. Morse, Am. Math. Soc. Colloq. Publ., 18, (1934).
[2] A. I. Fet, DAN, 66, No. 3, 347 (1949); 66, No. 3, 569 (1949).
[3] R. Bott, Ann. Math., 60, No. 2, 248 (1954).
[4] P. E. Conner, Proc. Nat. Acad. Sci. USA, 42, No. 6, 349 (1956).
[5] A. Borel, Ann. Math., 57, No. 1, 115 (1953).

Note: Figure translations are in progress. See original paper for figures.