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Abstract

Full Text

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ON FUNCTIONALLY CLOSED SPACES

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A completely regular space P is called **functionally closed** if, for every proper extension $\widetilde{P}$ of the space P, there exists a continuous real-valued function defined on P that cannot be extended* to $\widetilde{P}$. Such spaces, under the name of Q-spaces, were first considered by Hewitt ([1]).

We shall call an extension $\widetilde{P}$ of the space** P **regular** if every continuous function defined on P can be continuously extended to the extension $\widetilde{P}$. We shall call the **zero-set of a function** f, defined on the space P, the complete preimage $f^{-1}(0)$ of the number 0, and the **zero-set of the space** P a set that is the zero-set of some function.

Lemma 1. *Let $\widetilde{P}$ be a regular extension of the space P; for any continuous function f defined on the space P, its extension has as its zero-set the closure in $\widetilde{P}$ of the zero-set of the function f.*

Theorem 1. *In order that an extension $\widetilde{P}$ of the space P be regular, it is necessary that the closure of the intersection of any countable sequence of zero-sets of the space P coincide with the intersection of their closures in $\widetilde{P}$, and it is sufficient that, for any countable sequence of zero-sets with empty intersection, the sequence of closures of these sets in $\widetilde{P}$ also have empty intersection.*

Proof. Necessity. Let $\{f_i\}$ be a countable set of continuous functions defined on P, and suppose that everywhere $|f_i| \leq 1/2^i$. Then the function

$$\varphi = \sum_{i=1}^{\infty} f_i$$

is continuous, and the intersection of the zero-sets $N(f_i)$ of the functions f_i is its zero-set. From Lemma 1 we obtain

$$\widetilde{P}\left[\bigcap_i N(f_i)\right] = \bigcap_i \widetilde{P}[N(f_i)],$$

as required.

Sufficiency. Let I be the number line. To each point x of $\widetilde{P} \setminus P$ we assign the family $\mathfrak{B}(x)$ of all such zero-sets F of the space P that $x \in \widetilde{P}[F]$. For any continuous function f, defined on P, and for any point x of $\widetilde{P} \setminus P$, set

$$\tilde{f}(x) = \bigcap_{F \in \mathfrak{B}(x)} I[f(F)].$$

It turns out that the extension of the function f thus defined is continuous on P. The theorem is proved.

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* One says that a function f, defined on the space P, is **continuously extendable to a space** $S \supset P$ if there exists a continuous function $\tilde{f}$, defined on S, identically equal to f on P. The function $\tilde{f}$ is called an **extension** of the function f.

** By a space we shall always mean a completely regular space.

We shall call a nonempty maximal centered family of zero-sets of a space P **perfect** if the intersection of any of its countable subfamilies is nonempty. An extension $\widetilde{P}$ of a space P will be called a ϑ**-extension** if: 1) $\widetilde{P}$ is a regular extension of the space P; 2) $\widetilde{P}$ is functionally closed.

For every space P, denote by $\Omega(P)$ the family of all perfect families of this space that have empty intersection. For every zero-set F of the space P, denote by $\tau(F)$ the set consisting of all points of this set F and all perfect families from $\Omega(P)$ that contain the set F as an element.

Theorem 2. *The set ϑP, consisting of all points of the space P and of all elements of the family $\Omega(P)$, is a ϑ-extension of the space P, if one defines in it a topology by taking as closed sets all possible intersections of sets of the form $\tau(F)$.*

Proof. The fact that the extension ϑP of the space P so obtained is regular follows directly from Theorem 1. The functional closedness of the extension ϑP follows directly from the sufficiency of the following (Hewitt' s ([1])) condition for functional closedness:

Theorem 3. *In order that a space P be functionally closed, it is necessary and sufficient that every perfect family of the space P have nonempty intersection.*

Proof. The sufficiency of the stated condition again follows from Theorem 1, if one first observes that for every point x of $\widetilde{P} \setminus P$, where $\widetilde{P}$ is an arbitrary regular extension of the space P, the family $\mathfrak{B}(x)$ is a nonempty maximal centered family of zero-sets of the space P. The necessity of the condition follows directly from Theorem 2.

From Theorem 3 there follows directly:

Theorem 4. *For every space P there exists a unique (up to a topological mapping identical on P) ϑ-extension.*

We next clarify the connection between the notion of functional closedness and **topological completeness** in the sense of Dieudonné ([2]). The latter is based on the notion of a **uniform space** and a **uniform structure** in the sense of A. Weil ([3]). For our purposes it is more convenient to use the definition of a uniform structure as a certain system of coverings, equivalent to the original one ([4]), since we shall need the following completeness criterion for a uniform space, found by Yu. M. Smirnov:

A system ξ of subsets of the space P_Σ with covering structure Σ is called a Σ-**system** if, for every covering γ of the structure Σ, there is in the system ξ a set A contained in some element of the covering γ. The space P is complete relative to the structure Σ (i.e. the uniform space P_Σ is complete) if and only if every centered closed Σ-system has nonempty intersection (([5]), p. 431).

Consider the following systems of coverings of the space P: σ_0—the system of countable open normal* locally finite coverings; σ_0'—the system of countable open normal coverings; σ_1—the system of open normal locally finite coverings. Each of the systems considered generates a system Σ_0, respectively Σ_0' and Σ_1, consisting by definition of all those coverings of the space P in each—

* A covering γ, consisting of sets Γ_λ of the space P, is called **normal** if for each Γ_λ one can choose a set $A_\lambda \subseteq \Gamma_\lambda$, functionally separated from $P \setminus \Gamma_\lambda$, such that $\bigcup_\lambda A_\lambda = P$.

each of which contains some cover of the system under consideration.

Lemma 2. The systems Σ_0, Σ_0', and Σ_1 are uniform structures of the space P, and $\Sigma_0 = \Sigma_0'$.

Theorem 5. Every functionally closed space P is complete with respect to the structure Σ_1.

The proof is based on the fact that, if the uniform space P_{Σ_1} is not complete, then the space of its completion turns out to be a proper extension of the space P, distinct from P itself.

Theorem 6. Every space complete with respect to the structure Σ_0 is functionally closed.

Proof. Let ξ be a perfect family of the space P, complete with respect to the structure Σ_0. Since ξ turns out to be a centered closed Σ-system, its intersection is nonempty by the stated completeness criterion of Yu. M. Smirnov. By Theorem 3 the space P is functionally closed, as was required to prove.

Theorem 7. In order that a space be functionally closed, it is necessary and sufficient that it be complete with respect to the structure Σ_0.

We shall call a cardinal number $\mathfrak{m}$ ***F*-regular** if the discrete space of cardinality $\mathfrak{m}$ is functionally closed.

Theorem 8. Every normal space of F-regular cardinality, complete with respect to the structure Σ_1, is functionally closed*.

For normal spaces the structure Σ_1 is the maximal** structure.

Hence it follows:

Theorem 9. In order that a normal space of attainable cardinality*** be functionally closed, it is necessary and sufficient that it be topologically complete (i.e. complete with respect to the maximal uniform structure Σ_1).

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CITED LITERATURE

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[2] J. Dieudonné, Ann. Sci. de l' école norm. sup., 56, 4, 277 (1939).
[3] A. Weil, Actualités scientifiques et industrielles, 551, Sur les espaces à structure uniforme et sur la topologie générale, 1938.
[4] Yu. M. Smirnov, Mat. sborn., 31, 563 (1952).
[5] Yu. M. Smirnov, Tr. Mosk. matem. obshch., 4, 421 (1955).
[6] M. Katetov, Fund. Math., 38, 73 (1951).

* Cf. Katetov' s theorem (([6]), p. 82).
** A uniform structure of covers is **maximal** if it contains every other uniform structure of the space P as a substructure.
*** A cardinal number is attainable if it is less than the first unattainable cardinal number. A cardinal number $n > \aleph$ is unattainable if it cannot be represented as a sum of powers $2^{\mathfrak{k}_\lambda}$, the number of which is less than n.

Note: Figure translations are in progress. See original paper for figures.