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Abstract

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ON SOME QUASILINEAR PARABOLIC SYSTEMS

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1. In the present note, for systems of the form

$$\frac{\partial^2 u_i}{\partial x^2} = \frac{\partial u_i}{\partial t} + \sum_{j=1}^{N} b_{ij}(x, t, u_1, \dots, u_N)\frac{\partial u_j}{\partial x} + \sum_{j=1}^{N} c_{ij}(x, t, u_1, \dots, u_N)u_j + \\ + f_i(x, t), \qquad i = 1, \dots, N, \tag{1}$$

the existence of a solution of the Cauchy problem and of the first boundary-value problem in the rectangle $R_T\{x_1 \le x \le x_2,\ 0 \le t \le T\}$ is proved with boundary conditions

$$u_i(x, 0) = u_i^{(0)}(x), \qquad u_i(x_j, t) = u_i^{(j)}(t), \qquad i = 1, \dots, N, \quad j = 1, 2. \tag{2}$$

For the system

$$\varepsilon\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial v}{\partial t}, \qquad 0 = \frac{\partial v}{\partial t} + v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}, \tag{3}$$

which occurs in mechanics, the existence of a solution of the Cauchy problem and of the boundary-value problem in R_T is proved with conditions

$$u(x, 0) = u_0(x), \qquad u(x_1, t) = u(x_2, t) = 0, \qquad v(x, 0) = v_0(x). \tag{4}$$

2. **Theorem 1.** *A solution of problem (1), (2) in R_T, continuous in R_T, whose derivatives occurring in the system are bounded in R_T and continuous inside R_T, exists and is unique if the following assumptions are satisfied:*

1) For $(x, t) \in R_T$ and for arbitrary u_k,

$$|b_{ij}(x, t, u_1, \dots, u_N)| < B, \qquad \sum c_{ij}(x, t, u_1, \dots, u_N)\lambda_i\lambda_j \ge c\sum \lambda_i^2,$$

where B, c are constants.

2) For $(x,t) \in R_T$ and

$$\sum u_i^2 < M = \max\left\{\max_{\substack{(x,t)\in R_T\\ j=0,1,2}} \sum (u_i^{(j)})^2 e^{2CT}, \frac{NF^2}{C} e^{2CT}\right\}, \tag{5}$$

where $C > B^2N^2/2 + 2|c| + 1$, $F = \max|f_i(x,t)|$; the coefficients b_{ij}, their first-order derivatives and second derivatives with respect to x and u_k are continuous; the derivatives

$$\begin{gathered}\frac{\partial^2 b_{ij}}{\partial x\,\partial t}, \quad \frac{\partial^2 b_{ij}}{\partial u_k\,\partial t}, \quad \frac{\partial^3 b_{ij}}{\partial x^2 \partial t}, \quad \frac{\partial^3 b_{ij}}{\partial x\,\partial t\,\partial u_k}, \\ \frac{\partial^3 b_{ij}}{\partial x\,\partial u_k\,\partial u_l}, \quad \frac{\partial^3 b_{ij}}{\partial t\,\partial u_k\,\partial u_l}, \quad \frac{\partial^3 b_{ij}}{\partial x^2 \partial u_k}, \quad \frac{\partial^3 b_{ij}}{\partial u_k\,\partial u_l\,\partial u_m}\end{gathered} \tag{6}$$

are continuous inside R_T. The functions c_{ij} and f_i possess the same smoothness.

3) The functions $u_i^{(0)}(x)$ have second derivatives, bounded for $x_1 \le x \le x_2$ and continuous for $x_1 < x < x_2$; the functions $u_i^{(1)}(t)$, $u_i^{(2)}(t)$ have continuous first derivatives.

4) The compatibility conditions are satisfied

$$u_i^{(0)}(x_j) = u_i^{(j)}(0), \qquad i = 1, \dots, N, \quad j = 1, 2. \tag{7}$$

Remark. The derivatives $\partial u_i/\partial x$ of the solution obtained, for each fixed t, have limits as $x \to x_1$ and $x \to x_2$.

Theorem 1 is proved by means of Rothe' s method ([1]) and estimates analogous to the estimates in ([2,3]).

Using Theorem 1, one can prove the solvability of the Cauchy problem for the system (1).

Theorem 2. *A solution $u = \{u_1, \dots, u_N\}$ of the system (1), satisfying the initial conditions $u_i(x,0) = u_i^{(0)}(x)$, $i = 1, \dots, N$, continuous in the strip $S_T\{-\infty < x < \infty,\ 0 \le t \le T\}$, whose derivatives entering the system are continuous inside S_T, exists and is unique if the following assumptions are satisfied:*

1) *For $(x,t) \in S_T$, $|f_i(x,t)| < F$.*

2) *For $(x,t) \in S_T$ and arbitrary u_k, $|b_{ij}(x,t,u)| < B$, $\sum c_{ij}(x,t,u)\lambda_i\lambda_j \ge c\sum \lambda_i^2$; B, F, c are constants.*

3) *For $(x,t) \in S_T$ and*

$$\sum u_i^2 \leq M_0 = \max\left\{\max \sum \left(u_i^{(0)}\right)^2 e^{2CT},\ \frac{NF^2}{C}e^{2CT}\right\},$$

where $C > N^2B^2/2 + 2|c| + 1$, *the coefficients* b_{ij}, c_{ij}, f_i, *their first partial derivatives and second derivatives with respect to* x *and* u_k *are continuous and bounded, and the derivatives (6) and the same derivatives of* c_{ij} *and* f_i *are continuous inside* S_T.

4) *The functions* $u_i^{(0)}(x)$ *have two continuous and bounded derivatives.*

The solution of the Cauchy problem is obtained as the limit of solutions of the first boundary-value problem in an expanding sequence of rectangles. In proving the existence of this limit, the estimates obtained for the solutions of the first boundary-value problem and their derivatives are used.

Applying Theorem 1, inequality (5), and estimates for the derivatives of solutions of the first boundary-value problem, one can prove the theorem on existence of a solution of problem (1), (2), for sufficiently small T, for arbitrary systems of the form (1) with sufficiently smooth coefficients.

Theorem 3. *Suppose the following assumptions are satisfied:*

1) *In the whole space* $\{u_1, \dots, u_N\}$, *except for a certain closed set* S *containing at least one point all of whose coordinates are finite, the coefficients* b_{ij}, c_{ij}, f_i *satisfy the smoothness conditions specified in item 2) of Theorem 1, and the inequality* $\sum c_{ij}\lambda_i\lambda_j \geq c\sum \lambda_i^2$ *holds.*

2) *The functions (2) do not take values belonging to* S, *and satisfy condition 3) of Theorem 1.*

3) *The compatibility conditions (7) are satisfied.*

Then the solution of problem (1), (2), possessing all the properties indicated in Theorem 1 and satisfying the condition $\rho(u, S) \geq \rho_1 > 0$, *where* ρ_1 *is a constant for which the inequality* $\rho_1 < \rho(u_i^{(j)}, S) = \rho_0$ *is valid, exists and is unique for* $t \leq \tau(\rho_1, \rho_0)$, *where* τ *is a nonincreasing function of the first argument and a nondecreasing function of the second argument.*

Remark. Theorem 3 is also valid in the case when the set S contains no point all of whose coordinates are finite, under the corresponding definition of the distance from a point with finite coordinates to the set S.

Theorem 4. *Suppose all the conditions of Theorem 3 are satisfied and, in addition, suppose that for all solutions of (1), (2) the a priori estimate*

$$\rho(u, S) \geq a > 0. \tag{8}$$

is valid.

Then the solution of problem (1), (2) exists in the rectangle R_T for any T.

Proof. According to Theorem 3, the desired solution exists for $t \leqslant \tau(a/2, a)$. Suppose the solution exists for $t \leqslant \tau_1$. The solution can be continued, by virtue of Theorem 3, by solving the first boundary-value problem with the same boundary data and with initial data at $t = \tau_1$. The height of the rectangle to which the solution can be continued, by virtue of estimate (8) and the estimates for the derivatives of the solutions u_i entering the system, does not depend on τ_1. Therefore the solution exists everywhere in R_T.

3. Applying the theorems proved, one can prove the existence of a solution of the first boundary-value problem and of the Cauchy problem for the system

$$\varepsilon \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} - \frac{\partial \varphi(v)}{\partial x}, \qquad \varepsilon \frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} - \frac{\partial u}{\partial x}, \tag{9}$$

where the function $\varphi(v)$ is defined and four times continuously differentiable for $v > v_0 \geqslant -\infty$; the conditions $\varphi'(v) > 0$, $\varphi''(v) \leqslant 0$ are fulfilled, and, as $v \to v_0$,

$$F(v) = \int \sqrt{\varphi'(v)}\, dv \to -\infty.$$

Consider the first boundary-value problem for system (9); in doing so we shall assume that $v(x, 0)$, $v(x_1, t)$, $v(x_2, t) > v_0$.

According to Theorem 4, in order to prove the existence of a solution it is sufficient to establish that, if u, v is a solution of the stated boundary-value problem, then $v \geqslant v_1 > v_0$.

Since $\varphi'(v) > 0$, the first-order system obtained from system (9) for $\varepsilon = 0$ is hyperbolic. (For $\varphi(v) = -1/2v^2$ this system describes the motion of "shallow water," as well as one-dimensional isentropic motion of a gas for $c_p/c_v = 2$ in Lagrangian coordinates. System (3) for $\varepsilon = 0$ describes the same motions in Eulerian coordinates.)

The first-order system can be brought to the form

$$\frac{\partial f^+}{\partial t} + \sqrt{\varphi'(v)}\,\frac{\partial f^+}{\partial x} = 0, \qquad \frac{\partial f^-}{\partial t} - \sqrt{\varphi'(v)}\,\frac{\partial f^-}{\partial x} = 0,$$

where f^+ and f^- are new unknown functions (Riemann invariants):

$$f^+ = -F(v) + u, \qquad f^- = -F(v) - u.$$

If in system (9) one also passes from the functions u and v to the functions f^+ and f^-, it takes the form

$$\varepsilon \frac{\partial^2 f^+}{\partial x^2} - \frac{\partial f^+}{\partial t} = \sqrt{\varphi'(v)}\,\frac{\partial f^+}{\partial x} + \varepsilon \frac{\partial^2 f^+}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2;$$

$$\varepsilon \frac{\partial^2 f^-}{\partial x^2} - \frac{\partial f^-}{\partial t} = -\sqrt{\varphi'(v)}\,\frac{\partial f^-}{\partial x} + \varepsilon \frac{\partial^2 f^-}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2. \tag{10}$$

Since $\varphi''(v) \leqslant 0$, we have $\partial^2 f^+/\partial v^2 \geqslant 0$, $\partial^2 f^-/\partial v^2 \geqslant 0$, so that from equations (10) there follow the inequalities

$$f^+ \leqslant M^+ = \max_{x=x_1,x_2,\, t=0} f^+, \qquad f^- \leqslant M^- = \max_{x=x_1,x_2,\, t=0} f^-. \tag{11}$$

We have

$$F(v) = -\frac{f^+ + f^-}{2} \geqslant -\frac{M^+ + M^-}{2}, \tag{12}$$

and since $F'(v) = \sqrt{\varphi'(v)} > 0$, it follows from (12) that $v \geqslant v_1$. The inequality $v_1 > v_0$ follows from the monotonicity of $F(v)$ and from the fact that $F(v_0) = -\infty$. The existence of a solution of the first boundary-value problem is proved.

Using solutions of the first boundary-value problem, one can construct a solution of the Cauchy problem for system (9).

Theorem 5. In the strip S_T there exists a unique solution of system (9), satisfying the conditions

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x)$$

and possessing all the properties indicated in Theorem 2, if the following assumptions are fulfilled:

1) The function $\varphi(v)$ is defined and four times continuously differentiable for $v > v_0 \geq -\infty$, with $\varphi'(v) > 0$, $\varphi''(v) \leq 0$, $F(v) \to -\infty$ as $v \to v_0$.
2) $v_0(x) \geq v_1 > v_0$.
3) The functions u_0 and v_0 are twice continuously differentiable and bounded together with their first and second derivatives for $-\infty < x < \infty$.

The indicated method for obtaining an a priori estimate for the solution of the first boundary-value problem is applicable to a whole series of systems of the form (1) with two spatial variables.

4. Consider problem (3), (4). The substitution

$$w = \int_{x_1}^{x} v\,dx, \qquad z = \varepsilon u - w$$

brings system (3) into the form

$$\varepsilon \frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t} + \frac{w+z}{\varepsilon}\frac{\partial z}{\partial x}, \qquad 0 = \frac{\partial w}{\partial t} + \frac{w+z}{\varepsilon}\frac{\partial w}{\partial x}. \tag{13}$$

The initial and boundary values for w and z are determined from conditions (4); for $x = x_1,\ x = x_2$ the functions w and z are constant. Obviously,

$$\max|z| \leq \max|z_0|, \qquad \max|w| \leq \max|w_0|.$$

It can be proved that the first boundary-value problem for system (13) has a solution for

$$T < \frac{\varepsilon \ln 2}{4\max(|\partial w_0/\partial x| + |\partial z_0/\partial x|)}.$$

To prove the existence of a solution of the problem for arbitrary T, it is necessary to establish a priori estimates for $|\partial w/\partial x| = |w_x|$, $|\partial z/\partial x| = |z_x|$. For $x = x_1,\ x = x_2$ the function z_x is estimated in the same way as the analogous derivative in (2). Differentiating the second of equations (13) with respect to x, we obtain

$$0 = \frac{\partial w_x}{\partial t} + \frac{w+z}{\varepsilon}\frac{\partial w_x}{\partial x} + \frac{w_x + z_x}{\varepsilon} w_x,$$

whence it follows that at the point of maximum of $|w_x|$ the estimate $|w_x| \leq \max|z_x|$ is valid. To estimate $\max|z_x|$ in the case when this maximum is attained inside R_T or at $t = T$, one makes the substitution

$$z = \varphi(\bar z) = k\int_0^{\bar z} e^{-s^2/2}\,ds, \qquad k = \frac{2}{\sqrt{\pi}}\max|z|.$$

Then

$$L\bar z_x = \frac{\partial^2 \bar z_x}{\partial x^2} - \frac{\partial \bar z_x}{\partial t} - \frac{w+z}{\varepsilon}\frac{\partial \bar z_x}{\partial x}$$

$$= -\left(\frac{\varphi''}{\varphi'}\right)' \bar z_x^3 - 2\frac{\varphi''}{\varphi'}\frac{\partial \bar z_x}{\partial x}\bar z_x + \frac{w_x + \bar z_x \varphi'(\bar z)}{\varepsilon}\bar z_x.$$

At the point of maximum of $|\bar{z}_x|$,

$$\bar{z}_x L \bar{z}_x \geq \varepsilon \bar{z}_x^4 - 2\bar{z}_x^3 k/\varepsilon,$$

whence $|\bar{z}_x| < 2k/\varepsilon^2$. From the boundedness of $|\bar{z}_x|$ follows the boundedness of $|z_x|$. The solution of the Cauchy problem for system (3) is obtained as the limit of the solutions of problems (3), (4) as $x_1 \to -\infty,\ x_2 \to \infty$.

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Note: Figure translations are in progress. See original paper for figures.