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G. K. LEBED’

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Abstract

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G. K. LEBED’

INEQUALITIES FOR POLYNOMIALS AND THEIR DERIVATIVES

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In this note we give several theorems that provide estimates for the derivatives of a trigonometric polynomial

$$T_n(\theta) = \sum_{k=0}^{n} (a_k \cos k\theta + b_k \sin k\theta) \tag{1}$$

or of an algebraic polynomial

$$P_n(x) = \sum_{k=0}^{n} a_k x^k. \tag{2}$$

Theorem 1. If a trigonometric polynomial (1) of degree not exceeding n satisfies, for all θ, the inequality $|T_n(\theta)| \leq t_n(\theta)$, where $t_n(\theta)$ is a nonnegative function having a continuous derivative $t_n^{(r)}(\theta)$ of order r $(r = 0, 1)$ with modulus of continuity

$$\omega_r(h) = \sup_{|\theta_1 - \theta_2| \leq h} \left| t_n^{(r)}(\theta_1) - t_n^{(r)}(\theta_2) \right| \quad (\theta_1, \theta_2 \in [-\pi, \pi]),$$

then

$$\left| T_n^{(k)}(\theta) \right| \leq (An)^k \left\{ t_n(\theta) + k \frac{\omega_r(1/n)}{n^2} \right\} \quad (n, k = 1, 2, \dots,\ r = 0, 1), \tag{3}$$

where A is a constant independent of T_n and k.

In particular, when $t_n(\theta) = \text{const}$, $k = 1$, and $T_n(\theta)$ is an even trigonometric polynomial, inequality (3) becomes the well-known inequality for algebraic polynomials (([1]), pp. 72-75; ([2]), p. 27), but with a cruder constant. If $t_n(\theta)$ has a bounded second discontinuous derivative, then $\omega_1(t) = O(t)$, and therefore the remainder term $k\frac{\omega_r(1/n)}{n^r}$ on the right-hand side of (3) will have order $O(n^{-2})$.

Let us note in this connection that if we improve the properties of the function $t_n(\theta)$, for example, require that it have a bounded third derivative, this circumstance does not lead to an improvement in the order of the remainder term. It still remains equal to $O(n^{-2})$, as is easily verified from the example $T_n(\theta) = \sin^2 n\theta \le n^2 \sin^2 \theta \le t_n(\theta)$.

Definition. We shall say that a function $\omega(t)$ satisfies condition A_α^β if the following properties hold for it:

1) $\omega(t) > 0,\ t > 0;$

2) $\omega(t_1) \le c\omega(t_2)\ \left(0 \le t_1 \le t_2 \le \dfrac{2}{n^\alpha},\ 0 \le \alpha\right);$

3) $\dfrac{\omega(t_2)}{t_2^\beta} \le c\,\dfrac{\omega(t_1)}{t_1^\beta}\ \left(0 < t_1 \le t_2 \le \dfrac{2}{n^\alpha},\ 0 \le \beta\right),$

where c is a constant independent of t_1 and t_2.

Theorem 2. If, for the trigonometric polynomial $T_n(\theta)$, the inequality

$$|T_n(\theta)| \le \omega\left(\frac{|\sin\theta|}{n^\alpha} + \frac{1}{n^{1+\alpha}}\right) \qquad (0 \le \alpha),$$

holds, where $\omega(t)$ satisfies condition A_α^β, then

$$\left|T_n^{(k)}(\theta)\right| \le (An)^k \omega\left(\frac{|\sin\theta|}{n^\alpha} + \frac{1}{n^{1+\alpha}}\right) \qquad (n, k = 1, 2, \dots),$$

where A is a constant independent of $T_n(\theta)$ and k.

Theorem 3. If the function $\omega(t)$ satisfies condition A_α^β, where $\alpha = 1,\ 0 \le \beta \le 1$, then for any algebraic polynomial $P_n(x)$ and any r, p, p' satisfying the inequalities $1 \le p \le p' \le \infty$, one has

$$\left\|\frac{P_n(x)\delta^{r-1/p'}(x,n)}{\omega[\delta(x,n)/n]}\right\|_{L_{p'}(a,b)} \le A\left(\frac{2n}{b-a}\right)^{1/p-1/p'} \left\|\frac{P_n(x)\delta^{r-1/p}(x,n)}{\omega[\delta(x,n)/n]}\right\|_{L_p(a,b)},$$

where

$$\delta(u,\nu) = \frac{2\sqrt{(b-u)(u-a)}}{b-a} + \frac{1}{\nu}; \qquad A = A(r)$$

is a constant depending only on r, and

$$\|\varphi\|_{L_p(a,b)} = \left(\int_a^b |\varphi(x)|^p dx\right)^{1/p}.$$

For nonperiodic functions this inequality is an analogue of the inequality of S. M. Nikol' skii (^{3}) for trigonometric polynomials or entire functions of finite degree.

In particular, if in it we put $\omega(t) \equiv 1$, $r = 1/p$, $p' = \infty$, then we obtain the inequality

$$|P_n(x)| \leq \frac{A}{\delta^{1/p}(x,n)} \left(\frac{2n}{b-a}\right)^{1/p} \|P_n\|_{L_p(a,b)}, \qquad 1 \leq p \leq \infty,$$

which is a strengthening of Jackson' s inequality (^{4})

$$|P_n(x)| \leq K(a,b) n^{2/p} \|P_n\|_{L_p(a,b)}.$$

Theorem 4. Under the assumptions on the function $\omega(t)$ imposed in Theorem 3, for every algebraic polynomial $P_n(x)$ of degree not exceeding n the inequality

$$\left\| \frac{P_n^{(k)} \delta^{k+r}(x,n)}{\omega[\delta(x,n)/n]} \right\|_{L_p(a,b)} \leq \left(A \frac{2a}{b-a}\right)^k \left\| \frac{P_n(x) \delta^r(x,n)}{\omega[\delta(x,n)/n]} \right\|_{L_p(a,b)}, \tag{4}$$

$$(n, k = 1, 2, \dots; \quad 1 \leq p \leq \infty),$$

holds, where $A = A(r)$ is a constant depending only on r (r arbitrary).

In particular, if in (4) we put $k = 1$, $\omega(t) \equiv 1$, $r = -\rho$ and $p = \infty$, then we obtain the inequality of V. K. Dzyadyk (^{5})*

$$\left| \frac{P_n'(x)}{\delta^{\rho-1}(x,n)} \right| \leq An \max_{a \leq x \leq b} \left| \frac{P_n(x)}{\delta^\rho(x,n)} \right|.$$

* This inequality was obtained by us by another method independently of Dzyadyk and was reported at the seminar on the theory of approximation of functions at the V. A. Steklov Mathematical Institute of the Academy of Sciences of the USSR at the end of 1956.

If in (4) we put $\omega(t) \equiv 1$ and $r = 0$, then we obtain the inequality

$$\|P_n^{(k)}(x) \delta^k(x,n)\|_{L_p(a,b)} \leq \left(\frac{An}{b-a}\right)^k \|P_n(x)\|_{L_p(a,b)},$$

which is obviously stronger than the corresponding inequality of N. K. Bari (6)

$$\|P_n^{(k)}\|_{L_p(a,b)} \leq cn^{2k}\|P_n\|_{L_p(a,b)}.$$

In proving the theorems stated above, we used the fact that a trigonometric polynomial $T_n(\theta)$ of order n can be represented (in infinitely many ways) in the form of the integral

$$T_n(\theta) = \frac{1}{\pi}\int_0^{2\pi} K_m(u)T_n(\theta+u)\,du \qquad (m \leq n),$$

where $K_m(u)$ has the form

$$K_m(u) = \frac{1}{2} + \sum_{k=1}^{n} \cos ku + \sum_{k=n+1}^{m} (a_k \cos ku + b_k \sin ku),$$

and the coefficients a_k and b_k may be arbitrary. Then

$$T_n'(\theta) = \frac{1}{\pi}\int_0^{2\pi} K_m'(u)T_n(\theta+u)\,du.$$

To obtain our inequalities we used this representation of the derivative $T_n'(\theta)$, each time choosing the coefficients a_k and b_k in the appropriate way.

In these investigations the following lemma is also used:

Lemma. Let $P_n(x)$ be an algebraic polynomial of degree $\leq n$, and suppose that on the interval

$$\left[-1+\frac{c^2}{n^2},\ 1-\frac{c^2}{n^2}\right]$$

the inequality

$$|P_n(x)| \leq \left(\sqrt{1-x^2}+\frac{1}{n}\right)^s \omega\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)$$

holds. Then, if ω satisfies the conditions of Theorem 3, the inequality

$$|P_n(x)| \leq A\left(\sqrt{1-x^2}+\frac{1}{n}\right)^s \omega\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right),$$

where A is some constant ≥ 1, independent of n and x, holds on the entire interval $[-1, 1]$.

This note arose as a result of work on a Candidate dissertation carried out under the supervision of Prof. S. M. Nikol' skii.

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Note: Figure translations are in progress. See original paper for figures.