Mathematics

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Abstract

Full Text

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ON SOME ESTIMATES IN THE THEORY OF TOEPLITZ FORMS AND ORTHOGONAL POLYNOMIALS

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1. Consider the Toeplitz forms

$$T_n = \sum_{i,k=0}^{n} c_{i-k} x_i \bar{x}_k, \qquad c_{-n} = \bar{c}_n, \qquad \Delta_n = |c_{i-k}|_0^n \quad (n = 0, 1, 2, \ldots); \tag{1}$$

they are positive definite if $\{\Delta_n\}_0^\infty > 0$. Introduce the notation

$$h_n = \frac{\Delta_{n+1}}{\Delta_n} \qquad (n = 0, 1, 2, \ldots). \tag{2}$$

It is not difficult to prove that $0 < h_{n+1} \leq h_n$ ([6]); therefore there exists the limit

$$\lim_{n\to\infty} h_n = h \geq 0.$$

We shall consider the quantity

$$\mu_n = h_n - h = h_n - \lim_{n\to\infty} h_n \qquad (n = 0, 1, 2, \ldots) \tag{3}$$

and indicate some estimates for it.

If one introduces the parameters (([1]), pp. 36–38)

$$a_n = \frac{(-1)^n}{\Delta_n} |c_{i-k+1}|_0^n, \qquad h_n = h_0 \prod_{k=0}^{n-1} \{1 - |a_k|^2\} \qquad (n = 0, 1, 2, \ldots), \tag{4}$$

then it is clear that the conditions $\{\Delta_n\}_0^\infty > 0$ are equivalent to the conditions $\{|a_n|\}_0^\infty < 1$, and the condition $h > 0$ is equivalent to the convergence of the series $\sum_{k=0}^\infty |a_k|^2$; consequently, when these conditions are fulfilled, we have the estimate of the quantity μ_n in terms of the parameters:

$$\frac{\mu_n}{h_n} = \left| 1 - \prod_{k=n}^{\infty} \{1 - |a_k|^2\} \right| \sim \sum_{k=n}^{\infty} |a_k|^2, \qquad \mu_n = O\left\{ \sum_{k=n}^{\infty} |a_k|^2 \right\}. \tag{5}$$

2. When the conditions $\{\Delta_n\}_0^\infty > 0$ are fulfilled, we have the representation

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta}\, d\sigma(\theta) = c_k \qquad (k = 0, 1, 2, ...), \tag{6}$$

where $\sigma(\theta)$ is a bounded nondecreasing function with an infinite set of points of increase; if one introduces the polynomials $\{P_n(z)\}$, orthonormal on the circle $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, with respect to the mass distribution $d\sigma(\theta)$, then the quantity h_n also has the meaning

$$h_n = \frac{1}{\alpha_n^2}, \qquad P_n(z) = \alpha_n z^n + \cdots \qquad (n = 0, 1, 2, ...), \tag{7}$$

and, thus,

$$\mu_n = \frac{1}{\alpha_n^2} - \frac{1}{\alpha^2}, \qquad \lim_{n\to\infty} \alpha_n = \alpha \leqslant \infty, \tag{8}$$

The conditions

$$h = \frac{1}{\alpha} > 0, \qquad \sum_{k=0}^{\infty} |a_k|^2 < \infty, \qquad \lg \sigma'(\theta) \in L_1 \tag{9}$$

are equivalent to one another by virtue of the relation ((7); (1), p. 38)

$$h = \lim_{n\to\infty} \frac{\Delta_{n+1}}{\Delta_n} = h_0 \prod_{k=0}^{\infty}\{1 - |a_k|^2\} = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \lg \sigma'(\theta)\, d\theta\right\}; \tag{10}$$

in what follows we shall assume that these conditions are satisfied; they, in turn, are equivalent to the condition ((1), p. 21)

$$\lim_{n\to\infty} P_n^*(z) = \pi(z) = \exp\left\{-\frac{1}{4\pi}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \lg \sigma'(\theta)\, d\theta\right\},$$

$$|z| < 1, \qquad P_n^*(z) = z^n \overline{P_n}\left(\frac{1}{z}\right), \tag{11}$$

and for $|z| \leqslant r < 1$ we have uniformly

$$|P_n^*(z) - \pi(z)| = o(\sqrt{\mu_n}), \tag{12}$$

i.e., the error of the asymptotic formula inside the disk $|z| < 1$ is estimated by means of μ_n. We have the growth estimate for the orthonormal polynomials:

$$|P_n(e^{i\theta})| = \max_{0\leqslant\theta<2\pi} |P_n(e^{i\theta})| \leqslant |\pi(re^{i\theta})|(C_1 + C_2\sqrt{n\mu_n}), \qquad r = 1 - \frac{1}{2n}. \quad (13)$$

The case $\mu_n = O(1/n)$ is especially interesting: then we may put $C_2 = 0$ in (13). If, moreover, the condition

$$\sigma(\theta_2) - \sigma(\theta_1) \geqslant m(\theta_2 - \theta_1), \qquad 0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi, \quad (14)$$

is satisfied, then in the closed disk $|z| \leqslant 1$ the estimate

$$|P_n(z)| \leqslant C_3 + C_4\sqrt{n\mu_n}, \qquad |z| \leqslant 1 \quad (15)$$

holds. If both conditions are satisfied, the entire orthonormal system is uniformly bounded in the closed disk ([2]).

If $\mu_n = o(1/n)$, then, under (14), at every point $z_0 = e^{i\theta_0}$ at which there exists the radial boundary value $\pi(e^{i\theta_0})$, we have

$$|P_n^*(z_0) - \pi(z_0)| \leqslant \lambda_n, \qquad \lambda_n = C_1^3\sqrt{n\mu_n} + C_2|\pi(z_0) - \pi(rz_0)|, \qquad r = 1 - \mu_n^{1/3} n^{-2/3}. \quad (16)$$

Under the more restrictive condition

$$\sum_{n=1}^{\infty} \sqrt{\frac{\mu_n}{n}} < \infty$$

the function $\sigma(\theta)$ is absolutely continuous on the whole interval $[0, 2\pi]$, the function $\pi(z)$ is continuous in the closed disk, and the asymptotic formula (16) holds with error estimate ([5])

$$\lambda_n = C \sum_{k=n+1}^{\infty} \sqrt{\frac{\mu_k}{k}}. \quad (17)$$

Let us indicate still other estimates expressed in terms of the quantity μ_n:

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{P_n^+(e^{i\theta})}{\pi(e^{i\theta})} - 1\right|^2 d\theta \leq C_1\mu_n; \qquad \frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^2\, d\sigma_1(\theta) \leq C_2\mu_n; \quad (18)$$

$\sigma_1(\theta)$ is the sum of the jump function and the singular component of the function $\sigma(\theta)$.

3. As we have shown, many estimates are connected with the quantity μ_n; therefore it is very important to find estimates for μ_n. From estimate (5) it is clear that μ_n

Table 1

	Conditions imposed on the weight $p(\theta)$, $0 \leq \theta \leq 2\pi$	Upper estimates for the quantity $\sqrt{\mu_n}$
I	$0 < m \leq p(\theta) \leq M$	$C_1\omega_2\left(\frac{1}{n}; p\right)$ or $C_2\sqrt{\omega_1\left(\frac{1}{n}; p\right)}$
II	$0 < m \leq p(\theta)$	$C_1\omega_4\left(\frac{1}{n}; \lg p\right) +$ $C_2\omega_2\left(\frac{1}{n}; p\right)$ or $C_3\sqrt{\omega_1\left(\frac{1}{n}; p\right)}$
III	$p(\theta) \in L_r,\ \frac{1}{p(\theta)} \in L_{r'}, \frac{1}{r} + \frac{1}{r'} = 1,\ r > 1$	$C_1\omega_{2r}\left(\frac{1}{n}; \sqrt{p}\right) +$ $C_2\omega_4\left(\frac{2}{n}; \lg p\right)$ or $C_3\sqrt{\omega_r\left(\frac{1}{n}; p\right)}$
IV	$\frac{1}{p(\theta)} \in L_1,\ p(\theta) \leq M$	$C_1\omega_4\left(\frac{1}{n}; \lg p\right) +$ $C_2\omega_2\left(\frac{1}{n}; \frac{1}{\sqrt{p}}\right)$ or $C_3\sqrt{\omega_1\left(\frac{1}{n}; \frac{1}{p}\right)}$
V	$\lim_{\delta \to 0} I(\delta) = 0.$	$C_1\sqrt{I\left(\frac{1}{n}\right)}$

can tend to zero arbitrarily slowly, since the parameters $\{a_n\}$ may be chosen quite arbitrarily, provided only that the conditions

$$|a_n| < 1, \qquad n = 0, 1, 2, \dots, \qquad \sum_{n=0}^{\infty} |a_n|^2 < \infty \tag{19}$$

are satisfied.

Assuming that the function $\sigma(\theta)$ is absolutely continuous on the whole interval $[0, 2\pi]$, and introducing the notation

$$\omega_r(\delta; f) = \sup_{|h|\le\delta} \|f(\theta+h) - f(\theta)\|_r, \qquad f \in L_r,$$

$$I(\delta) = \sup_{|h|\le\delta} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{|p(\theta+h) - p(\theta)|}{p(\theta)}\, d\theta \right\},$$

we indicate in Table 1 some estimates for the quantity μ_n, expressed in terms of the structural characteristics of the weight $p(\theta) = \sigma'(\theta)$.

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Note: Figure translations are in progress. See original paper for figures.