ON ONE CLASS OF CONTINUOUS MAPPINGS OF CERTAIN INFINITE-DIMENSIONAL SETS

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S. S. RYSHKOV

ON ONE CLASS OF CONTINUOUS MAPPINGS OF CERTAIN INFINITE-DIMENSIONAL SETS

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In this note an Λ-class of mappings of subsets of Hilbert space H is defined. This class is of interest, for example, in that for H-polyhedra homeomorphic in this class, the infinite homology groups ([1]) are isomorphic. Moreover, every mapping $\varphi : M \to H$ having the form $\lambda e + a$ (where λ is a real number; a is a continuous mapping of the set $M \subset H$ into a compact set; e is the identity mapping) belongs to the class Λ.

I. **Definition of the class** $\Lambda_{\mathfrak{B},\widetilde{\mathfrak{B}}}$. Let $\mathfrak{B} = \{f_1, f_2, \dots, f_n, \dots\}$ and $\widetilde{\mathfrak{B}} = \{\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_n, \dots\}$ be arbitrary orthonormal bases of Hilbert space H; let M be a subset of this space. We shall say that a continuous mapping $\varphi : M \to H$ **belongs to the class** $\Lambda_{\mathfrak{B},\widetilde{\mathfrak{B}}}$ if it has the following properties:

1) For every point $x \in M$ and numbers $r > 0$ and $\varepsilon > 0$, there exist numbers

$$\lambda_2 = \lambda_2(x) \geq \lambda_1 = \lambda_1(x) > 0$$

and

$$N = N(x, r, \varepsilon, \mathfrak{B}, \widetilde{\mathfrak{B}}),$$

such that for $n > N$ and $y \in M \cap O_r x$ the inequalities

$$\lambda_1 \rho_n(x, y) - \varepsilon \leq \tilde{\rho}_n(\varphi(x),$$

$$\varphi(y)) \leq \lambda_2 \rho_n(x, y) + \varepsilon,$$

are satisfied, where $\rho_n(x, y)$ is the distance between the projections of the points x and y onto the linear span ${}^nH$ of all vectors of the basis $\mathfrak{B}$, with the exception of the first n; $\tilde{\rho}_n(x, y)$ is defined similarly, only with $\mathfrak{B}$ replaced by $\widetilde{\mathfrak{B}}$.

2) For every point $x \in M$ and numbers $r > 0$ and $\varepsilon > 0$, there exists a number $N = N(x, r, \varepsilon, \mathfrak{B}, \widetilde{\mathfrak{B}})$ such that for any $n > N$ and

$$y \in M \cap (H_+^n + x) \cap O_r x,$$

where $O_r x$ is the open ball of radius r with center at the point x, the inequality

$$\rho(\varphi(y), \varphi(x) + \widetilde{H}^n_+) \leq \varepsilon$$

is satisfied

(here H^n_+ denotes the half-space formed by the vectors

$$b_1 f_1 + b_2 f_2 + \dots + b_{nf} n,$$

where $b_1, b_2, \dots, b_{n-1}$ are arbitrary and $b_n > 0$; the half-space $\widetilde{H}^n_+$ is defined analogously).

3) The preimage of a bounded set is bounded.

II. **Definition of the class Λ.** We shall say that a continuous mapping $\varphi : M \to H$ belongs to the class Λ if it satisfies conditions 2) and 3), as well as the following condition:

1′) There exist two bases

$$\mathfrak{B} = \{f_1, f_2, \dots, f_n, \dots\}$$

and

$$\widetilde{\mathfrak{B}} = \{\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_n, \dots\},$$

such that for any $x \in H$, $r > 0$, and $\varepsilon > 0$ one can

choose the numbers $\lambda_2 = \lambda_2(x)$, $\lambda_1 = \lambda_1(x)$ and $N = N(x, r, \varepsilon, \mathfrak{B}, \tilde{\mathfrak{B}})$, having the property that, for the plane R spanned by the vectors

$$f_1, f_2, \dots, f_N, \quad \sum_{i=N+1}^{\infty} a_1^i f_i, \ \dots, \ \sum_{i=N+1}^{\infty} a_k^i f_i,$$

where

$$\sum_{i=N+1}^{\infty} (a_l^i)^2 < \infty,$$

the inequality

$$\lambda_1 \rho(R + x, R + y) - \varepsilon \leq \rho(\tilde{R} + \varphi(x), \tilde{R} + \varphi(y)) < \lambda_2 \rho(R + x, R + y) + \varepsilon;$$

holds; here $y \in M \cap O_r x$; $\tilde{R}$ is the plane spanned by the vectors

$$\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_N, \quad \sum_{i=N+1}^{\infty} a_1^i \tilde{f}_i, \dots, \quad \sum_{i=N+1}^{\infty} a_k^i \tilde{f}_i.$$

Theorem. *If a mapping $\varphi : M \to H$ belongs to the class Λ, then for every basis $\mathfrak{B}$ there exists a basis $\tilde{\mathfrak{B}}$ such that $\varphi \in \Lambda_{\mathfrak{B},\tilde{\mathfrak{B}}}$.*

II. Lemmas, examples.

A. Let $\varphi \in \Lambda_{\mathfrak{A},\mathfrak{A}'}$ and $\psi \in \Lambda_{\mathfrak{A},\mathfrak{A}'}$; then also $\varphi + \psi \in \Lambda_{\mathfrak{A},\mathfrak{A}'}$.

B. Let $\varphi \in \Lambda_{\mathfrak{A}',\mathfrak{A}'}$ and $\psi \in \Lambda_{\mathfrak{A}',\mathfrak{A}''}$; then $\psi\varphi \in \Lambda_{\mathfrak{A},\mathfrak{A}''}$.

C. If $\varphi \in \Lambda_{\mathfrak{A},\mathfrak{B}}$ and the basis $\mathfrak{A}$ is comparable with the basis $\mathfrak{A}'$, while the basis $\mathfrak{B}$ is comparable with $\mathfrak{B}'$ ([1]), then $\varphi \in \Lambda_{\mathfrak{A}',\mathfrak{B}'}$.

By virtue of the last assertion, instead of $\Lambda_{\mathfrak{A},\mathfrak{B}}$ we may write $\Lambda_{\alpha,\beta}$, where α and β are the corresponding senses of orientation.

D. Any mapping $\varphi : M \to H$ of the form $\lambda e + a$ (where λ is a real number; a is a continuous mapping of the set M into a compact set; e is the identity mapping) belongs to the class $\Lambda_{\alpha,\alpha}$ for any sense of orientation α, and also to the class Λ.

E. Let $\tilde{\varphi}$ be some continuous positive function of a real variable; then the mapping $\varphi : H \to H$, given by the formula $\varphi(x) = x\tilde{\varphi}(\rho(0, x))$, belongs to the class $\Lambda_{\alpha,\alpha}$ for any sense of orientation α, and also belongs to the class Λ.

IV. Invariance theorems. Let K be a subcomplex of some triangulation of the Hilbert space H. The closure P of the body of the subcomplex K will be called an H-polyhedron. Note that, on the basis of the invariance theorem formulated in ([2]), one may speak of the homotopy groups of the H-polyhedron P, and not of the subcomplex defining it; we shall denote the homology groups of an H-polyhedron by ${}_rH(P)$.

Theorem 1. *Let P_1 and P_2 be two H-polyhedra; let $f : P_1 \to H$ and $g : P_2 \to H$ be mappings belonging, respectively, to the classes $\Lambda_{\alpha,\beta}$ and $\Lambda_{\beta,\alpha}$, such that $fg = e$ and $gf = e$, where e denotes the identity mappings. Then ${}_nH(P_1) \cong {}_nH(P_2)$ for any n. If, moreover, $f \in \Lambda$ and $g \in \Lambda$, then the isomorphism ${}_nH(P_1) \cong {}_nH(P_2)$ constructed from these mappings does not depend on the choice of the classes $\Lambda_{\alpha,\beta}$ and $\Lambda_{\beta,\alpha}$.*

Theorem 2. *Let F and F' be (closed) subsets of the space H; suppose there exist two systems of open sets O_i and O_i', for which the inclusions*

$$F \subset \ldots \subset O_{\eta_i}F \subset O_i \subset O_{\eta_{i-1}}F \subset O_{i-1} \subset \ldots \subset O_1 \subset H,$$

$$F' \subset \ldots \subset O_{\xi_i}F' \subset O_i' \subset O_{\xi_{i-1}}F' \subset O_{i-1}' \subset \ldots \subset O_1' \subset H,$$

hold, where ξ_i and η_i are numbers tending to zero as the index i increases, and

by $O_\delta M$ is denoted the δ-neighborhood of the set M. Let also $f : O_1 \to H$ and $g : O_1' \to H$ be mappings belonging, respectively, to the classes $\Lambda_{\alpha,\beta}$ and $\Lambda_{\beta,\alpha}$, and having the property that for each pair of sets F and F', O_1 and O_1', O_2 and O_2', etc., the mappings f and g are mutually inverse homeomorphisms.

Then

$$ {}_nH_{\text{vn}}(F) \cong {}_nH_{\text{vn}}(F'), $$

and, by means of the mappings f and g, a quite definite isomorphism of these groups is constructed. If, moreover, $f \in \Lambda$ and $g \in \Lambda$, then the constructed isomorphism does not depend on the choice of the classes $\Lambda_{\alpha,\beta}$ and $\Lambda_{\beta,\alpha}$. (For the definition of ${}_nH_{\text{vn}}(F)$, see ([2]).)

The following lemmas are central in the proof of these assertions.

Lemma 1. Let $\varphi : K \to H$ be a mapping of the complex K into the space H such that, on the preimage of some neighborhood $O_\rho y$ of some point $y \in H$, the mapping φ belongs to the class $\Lambda_{\alpha,\beta}$. Then, for any n-plane ${}^nR$ of defect n passing through the point y, there exists a completely continuous mapping $\varphi^* : K \to H$ such that:

a) $\varphi^*(K \setminus \varphi^{-1}(O_\rho y)) = 0$;

b) the complex K is mapped into a plane parallel to ${}^nR$;

c) $\operatorname{diam} \varphi^*(K) < 2\rho$;

d) $y \notin (\varphi + \varphi^*)({}^{r+1}K)$, where ${}^{r+1}K$ denotes the $(r+1)$-defective skeleton of the complex K.

Lemma 2. Let $\varphi : F \to H$ be a mapping of a closed subset $F \subset H$ into H. If some point $y \in H$ does not belong to the image of the set F, and $\varphi \in \Lambda_{\alpha,\beta}$, then there exists ρ such that the intersection $\varphi(F) \cap O_\rho y$ is empty.

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Moscow State University
named after M. V. Lomonosov

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References Cited

[1] V. G. Boltyanskii, DAN, **105**, No. 6 (1955).
[2] S. S. Ryshkov, DAN, **114**, No. 3 (1957).

Note: Figure translations are in progress. See original paper for figures.