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Abstract

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ON THE UNIQUENESS OF THE SOLUTION OF FRANKL' S PROBLEM FOR THE CHAPLYGIN EQUATION

(Presented by Academician M. A. Lavrent' ev, 19 IX 1956)

Our note ([1]) was devoted to Frankl' s problem ([2]) in the case of the Lavrent' ev equation.

Consider the Chaplygin equation

$$k(y)u_{xx} + u_{yy} = 0, \tag{1}$$

where

$$k(0) = 0, \quad k'(y) > 0, \quad k(-y) = -k(y). \tag{2}$$

In a mixed domain D, bounded by: a) the segment $A'A$ of the straight line $x = 0$, $-a \leq y \leq a$; b) the characteristic $A'B$ of equation (1), $B(a_1, 0)$, $a_1 > 0$; c) a chordal smooth arc σ with endpoints at B and A, lying in the half-plane $y > 0$, Frankl' s problem is posed as follows: *to find a regular solution of equation* (1) *in the domain* D, *continuous in the closed domain* $\overline{D}$ *and satisfying the boundary conditions:*

$$u\big|_{\sigma} = \varphi, \tag{3}$$

$$u_x\big|_{A'A} = 0, \tag{4}$$

$$u(0, y) - u(0, -y) = \Psi(y), \quad -a \leq y \leq a, \tag{5}$$

where φ and Ψ are prescribed functions.

The method of proving the uniqueness of the solution of this problem indicated in ([1]), under the condition that

$$dy/ds \geq 0, \tag{6}$$

where $x = x(s)$, $y = y(s)$ are parametric equations, and s is the length of the arc σ, measured from the point B, is applicable also in the case of equation (1).

Denote by $v(x, y)$ a function which, together with the desired solution $u(x, y)$, satisfies the system of equations

$$k(y)u_x - v_y = 0, \quad u_y + v_x = 0. \tag{7}$$

Under the condition that $v(0, 0) = 0$, the function $v(x, y)$ is uniquely determined by $u(x, y)$ from relations (7). In addition, we shall require that $u(x, y)$ satisfy conditions ensuring the continuity of $v(x, y)$ in the closed domain $\overline{D}$.

Along any closed rectifiable curve C lying in the domain D, by virtue of (7) we have

$$\int_C (ku^2 - v^2)\, dy + 2uv\, dx = 0. \tag{8}$$

The validity of equality (8) is evident also in the case when C coincides with the entire boundary of the domain D. Using this equality, it is easy to verify that,

that the homogeneous Frankl problem ($\varphi = \psi = 0$) has only the trivial solution.

Indeed, since by virtue of (4) we have $v(0, y) = 0$, then on the basis of (2), (3), and (5) we may write

$$\int_\sigma v^2 y_s\, ds + \int_{A'B} (\sqrt{-ku} - v)^2\, dy = 0. \tag{9}$$

By virtue of (2) and (6) we conclude that each term on the right-hand side of (9) is equal to zero. Consequently, on the portions of the arc σ where $dy/ds > 0$, we have $u = v = 0$. But in the elliptic part D_1 of the mixed domain D, the system (7) is elliptic. Therefore, on the basis of the results of Carleman [(3)] and I. N. Vekua [(4)], we conclude that $u = v = 0$ in the domain D_1. Consequently, we have

$$u(x, 0) = 0, \qquad u_y(x, 0) = 0, \qquad 0 \leq x \leq a_1. \tag{10}$$

On the other hand, it is well known (see, for example, [(5)]) that the singular Cauchy problem (10), under fairly general restrictions on the function $k(y)$, cannot have a solution different from zero. Thus, $u = 0$ everywhere in the domain D.

The restriction (6), imposed on σ, is caused only by the method of proof.

We note that the integral equation to which the proof of the existence of a solution of the Frankl problem in the case under consideration can be reduced, in its singular part, in fact differs little from the kernel obtained in [1] of the integral equation, but nevertheless it requires additional investigation.

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Note: Figure translations are in progress. See original paper for figures.