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Abstract

Full Text

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ON THE REDUCTION OF MIXED BOUNDARY-VALUE PROBLEMS TO THE RIEMANN BOUNDARY-VALUE PROBLEM

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In Koiter' s work ([1]), the Hopf and Wiener method was used in solving three mixed problems for the biharmonic equation in the strip $0 < y < 1$. The use of the theory of the Hopf and Wiener integral equation in solving these problems is not necessary: one may use the theory of the Riemann boundary-value problem. Below a method is proposed which makes it possible to reduce to the Riemann problem the mixed problems for one type of partial differential equations, in the investigation of which the Fourier transform is usually used, in a domain whose boundary consists of segments and rays parallel to the x-axis. The Hopf and Wiener method is applicable by no means to all of these problems. The essence of the method will be explained for the simple equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - h^2 u = 0, \qquad -\infty < x < \infty. \tag{1}$$

§ 1. Let us determine the character of the behavior of the function $u(x, y)$ in accordance with which a boundary-value problem will later be posed. This behavior depends on the class of solutions. Let the boundary of the domain consist of parts of the straight lines $y = y_k$, $k = 1, \dots, n$; $-\infty = y_0 < y_1 < \dots < y_n < y_{n+1} = +\infty$ (the disposition of these parts will be taken into account when the problem is posed).

We shall seek a solution in the class of functions satisfying the conditions

$$\int_{-\infty}^{\infty} \left| \frac{\partial^{\mu+\nu} u(x, y)}{\partial x^\mu \partial y^\nu} \right|^2 dx < \text{const}, \qquad y \neq y_k, \quad \mu + \nu \leq 2. \tag{2}$$

From the latter follow the conditions imposed on the Fourier transform $\tilde{u}$:

$$\int_{-\infty}^{\infty} \left| x^\mu \frac{\partial^\nu \widetilde{u(x, y)}}{\partial y^\nu} \right|^2 dx < \text{const}. \tag{3}$$

We transform equation (1) by Fourier in each of the domains $y_{k-1} < y < y_k$ and write the solutions of the ordinary differential equations obtained:

$$\widetilde{u(x,y)} = C_k(x)e^{-y\sqrt{x^2+h^2}} + D_k(x)e^{y\sqrt{x^2+h^2}}, \qquad y_{k-1} < y < y_k, \quad k = 1, \dots, n+1. \tag{4}$$

We establish the restrictions which, in view of conditions (3), must be imposed on the arbitrary functions C_k and D_k:

$$C_1(x) \equiv 0, \qquad D_{n+1}(x) \equiv 0, \qquad -\infty < x < \infty,$$

$$x^\mu C_k(x)e^{-y_{k-1}|x|} \in L_2(-\infty,\infty), \qquad x^\mu D_k(x)e^{y_k|x|} \in L_k(-\infty,\infty), \tag{5}$$

$$\mu = 0, 1, 2; \qquad k = 1, \dots, n+1.$$

Thus the solution $u(x,y)$ contains $2n$ arbitrary functions from the classes (5).

§ 2. Let us pose the problem:

$$\alpha_{jkp}u(x, y_k - 0) + \beta_{jkp}\frac{\partial}{\partial y}u(x, y_k - 0) + \gamma_{jkp}u(x, y_k + 0) +$$

$$+\delta_{jkp}\frac{\partial}{\partial y}u(x, y_k + 0) = f_{kp}(x), \qquad x_j < x < x_{j+1}, \tag{6}$$

$$k = 1, \dots, n; \qquad j = 0, \dots, m \; (m \geq 0); \qquad p = 1, 2; \qquad x_0 = -\infty, \quad x_{m+1} = +\infty.$$

Concerning the condition imposed on the constants $\alpha_{jkp}, \beta_{jkp}, \gamma_{jkp}$ and δ_{jkp}, it will be stated below.

If the segment $x_j < x < x_{j+1}$ of the straight line $y = y_k$ does not belong to the boundary, then we put $\alpha_{jk1} = \beta_{jk2} = -\gamma_{jk1} = -\delta_{jk2} = 1$, $\alpha_{jk2} = \beta_{jk1} = \gamma_{jk2} = \delta_{jk1} = 0$, $f_{kp}(x) \equiv 0$ for $x_j < x < x_{j+1}$. For the solvability of the problem it is necessary that the functions f_{kp}, f'_{kp} belong to $L_2(-\infty,\infty)$. Necessary and sufficient conditions for solvability can be derived from the solvability conditions of the Riemann problem obtained below.

In order to transform the conditions (6) by Fourier, we extend each of them to the entire x-axis, adding to the right-hand sides the differences $\varphi^+_{jkp}(x - x_{j+1}) - \varphi^-_{jkp}(x - x_j)$, where $\varphi^+_{mkp}(x) \equiv \varphi^-_{0kp}(x) \equiv 0$ for $-\infty < x < \infty$; $\varphi^+_{jkp}(x) \equiv 0$ for $x < 0$; $\varphi^-_{jkp}(x) \equiv 0$ for $x > 0$. It is not difficult to establish that the unknown functions $\varphi^\pm_{jkp}$ must belong to the class $L_2(-\infty,\infty)$.

As a result of the transformation we shall have the system

$$\alpha_{jkp}\left[C_k(x)e^{-y_k\sqrt{x^2+h^2}}+D_k(x)e^{y_k\sqrt{x^2+h^2}}\right]+\beta_{jkp}\sqrt{x^2+h^2}\left[-C_k(x)e^{-y_k\sqrt{x^2+h^2}}+\right.$$

$$\left.+D_k(x)e^{y_k\sqrt{x^2+h^2}}\right]+\gamma_{jkp}\left[C_{k+1}(x)e^{-y_k\sqrt{x^2+h^2}}+D_{k+1}(x)e^{y_k\sqrt{x^2+h^2}}\right]+$$

$$+\delta_{jkp}\sqrt{x^2+h^2}\left[-C_{k+1}(x)e^{-y_k\sqrt{x^2+h^2}}+D_{k+1}(x)e^{y_k\sqrt{x^2+h^2}}\right]=$$

$$=\tilde{f}_{kp}(x)+\widetilde{\varphi}^{+}_{jkp}(x)e^{ix_{j+1}x}-\widetilde{\varphi}^{-}_{jkp}(x)e^{ix_jx}, \tag{7}$$

$$-\infty<x<\infty;\qquad j=0,\dots,m;\qquad k=1,\dots,n;\qquad p=1,2,$$

where

$$C_{n+1}\equiv D_1\equiv\widetilde{\varphi}^{+}_{mkp}\equiv\widetilde{\varphi}^{-}_{0kp}\equiv 0,$$

and $\widetilde{\varphi}^{+}_{jkp}(x)$ and $\widetilde{\varphi}^{-}_{jkp}(x)$ are the unknown boundary values of the functions $\widetilde{\varphi}^{+}_{jkp}(z)$ and $\widetilde{\varphi}^{-}_{jkp}(z)$, analytic in the half-planes $\operatorname{Im} z>0$ and $\operatorname{Im} z<0$, respectively, and such that ([2], p. 170):

$$\int_{-\infty}^{\infty}\left|\widetilde{\varphi}^{\pm}_{jkp}(x+iy)\right|^2\,dx<\text{const}. \tag{8}$$

§ 3. Let the constants $\alpha_{jkp},\beta_{jkp},\gamma_{jkp}$ and δ_{jkp} be such that the system (7) can be solved with respect to the functions C_k and D_k. Eliminating these functions, we obtain a system of $2mn$ boundary conditions with $4mn$ unknown functions $\widetilde{\varphi}^{+}_{jkp}$ and $\widetilde{\varphi}^{-}_{jkp}$. Thus, a Riemann problem (5), (7), (8) has been obtained for a system of $2mn$ pairs of functions, equivalent to the original boundary-value problem (6). In a number of cases, in solving this problem one may use known results from the theory of the Riemann problem ([3-5] and others). If the numbers x_j, y_k form a countable set, then by the indicated method one can obtain a Riemann problem for an infinite system.

If in the boundary conditions (6), for $k=k_i,\ i=1,\dots,r-1$, the functions $u(x,y_k+0)$ and $\dfrac{\partial}{\partial y}u(x,y_k+0)$ occur separately from the functions $u(x,y_k-0)$ and $\dfrac{\partial}{\partial y}u(x,y_k-0)$, which will be the case, for example, when $\alpha_{jk_i1}=\beta_{jk_i1}=\gamma_{jk_i2}=\delta_{jk_i2}=0$, then the system (7) splits into r systems. In this case we have r independent boundary-value problems. Consequently, such domains as a half-plane or a strip are not excluded from consideration.

§ 4. Let us explain the general reasoning on concrete examples.

1) The domain is the plane with a cut $y = 0,\ x > 0$. In this case $n = m = 1,\ x_1 = y_1 = 0,\ \alpha_{011} = \beta_{012} = -\gamma_{011} = -\delta_{012} = 1,\ \alpha_{012} = \beta_{011} = \gamma_{012} = \delta_{011} = 0,\ f_{11}(x) = f_{12}(x) \equiv 0$ for $x < 0$. Prescribing on the boundary the values of the sought function: $u(x, -0) = f_{11}(x),\ u(x, +0) = f_{12}(x),\ x > 0$, after transforming the system we obtain

$$C_1(x) - D_2(x) = \tilde{f}_{11}(x) + \widetilde{\varphi}^{+}_{011}(x); \qquad C_1(x) = \tilde{f}_{11}(x) - \widetilde{\varphi}^{-}_{111}(x),$$

$$-\sqrt{x^2 + h^2}\, C_1(x) - \sqrt{x^2 + h^2}\, D_2(x) = \tilde{f}_{12}(x) + \widetilde{\varphi}^{+}_{012}(x),$$

$$D_2(x) = \tilde{f}_{12}(x) - \widetilde{\varphi}^{-}_{112}(x).$$

2) The domain is the half-plane $y > 0,\ n = k_1 = 1,\ y_1 = 0$. Posing the problem $u(x, +0) = f_{11}(x),\ x > 0;\ \dfrac{\partial}{\partial y} u(x, +0) = f_{11}(x),\ x < 0\ (m = 1,\ x_1 = 0)$, we obtain

$$-\sqrt{x^2 + h^2}\, C_1(x) = \tilde{f}_{11}(x) - \widetilde{\varphi}^{-}_{11}(x), \qquad C_1(x) = \tilde{f}_{11}(x) + \widetilde{\varphi}^{+}_{111}(x).$$

In these examples the systems are solved by quadratures.

§ 5. The study of the equation

$$\sum_{\mu=0}^{p} \sum_{\nu=0}^{q} a_{\mu\nu}(y) \frac{\partial^{\mu+\nu}}{\partial x^\mu \partial y^\nu} u(x, y) = 0,$$

where $a_{\mu\nu}(y)$ are piecewise-constant functions, is of an analogous character. Instead of equality (4) we shall have

$$\tilde{u}(x, y) = \sum_{\nu=1}^{q_k} C_{\nu k}(x) e^{\omega_{\nu k}(x) y} \quad \text{for} \quad y_{k-1} < y < y_k$$

(the case of distinct roots of the characteristic equations is taken). Denoting by $M^{+}_{\nu k}$ and $M^{-}_{\nu k}$ the sets of points for which $\operatorname{Re} \omega_{\nu k}(x) > 0$ and $\operatorname{Re} \omega_{\nu k}(x) < 0$, respectively, we obtain conditions analogous to conditions (5). In particular,

$$C_{\nu(n+1)}(x) \equiv 0 \quad \text{for } x \in M^{+}_{\nu 1}, \qquad C_{\nu 1}(x) \equiv 0 \quad \text{for } x \in M^{-}_{\nu(n+1)}.$$

Instead of the class (2) one may take another class of functions $u(x, y)$, in which the Fourier transform with respect to the variable x is defined in some sense. The method set forth is suitable for nonhomogeneous equations and for functions $a_{\mu,\nu}(y)$ of a more general form. Generalizations are possible in the direction of increasing the number of equations and the number of variables with respect to which the Fourier transform is carried out.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.