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Abstract

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ON NON-ANALYTIC SOLUTIONS OF THE GOURSAT PROBLEM FOR A SYSTEM OF DIFFERENTIAL EQUATIONS WITH TWO INDEPENDENT VARIABLES

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In the present note the following problem is considered (the Goursat problem): to find, in the whole plane (x, t) (or in some neighborhood of $(0, 0)$), a solution of the system of equations

$$\frac{\partial u_i}{\partial t} = \sum_{j=1}^{n} b_{ij} \frac{\partial u_j}{\partial x} + F_i(x, t), \qquad i = 1, \dots, n, \tag{1}$$

under the conditions

$$u_i(l_i) = \varphi_i(t); \qquad l_i \text{ is the straight line } x = \mu_i t,\ i = 1, \dots, n; \quad -\infty < t < \infty; \tag{2}$$

b_{ij}, μ_i are constants, $i, j = 1, \dots, n$; $F_i(x, t)$, $\varphi_i(t)$ are everywhere continuously differentiable; $F_i(x, t)$, as $|x| + |t| \to \infty$, and $\varphi_i(t)$, as $|t| \to \infty$, may grow no faster than some power of $|x| + |t|$ and $|t|$, respectively.

We shall call problem (1)–(2) **well posed** if:

1) for any sufficiently smooth $\varphi_i(t)$, $i = 1, \dots, n$, in the class of functions everywhere continuously differentiable there exists a unique solution;

2) for any $A' > 0$ there can be found $a > 0$, $A > 0$, such that, under a sufficiently small change of $\varphi_i(t)$, $i = 1, \dots, n$, together with their derivatives up to some order on the intervals $[-A, -a]$ and $[a, A]$, the solution of the problem in the disk $x^2 + t^2 \leq {A'}^2$ changes only slightly.

In the work of L. A. Mel' tser ([1]), certain sufficient conditions are considered for the well-posedness of the Goursat problem in the case when $u_i(x, t)$, $i = 1, \dots, n$, are prescribed on the axes OX and OT, and all characteristics of (1) lie in the second and fourth quadrants (in this case the very notion of well-posedness is narrower there).

In what follows we restrict ourselves only to systems (1) that are hyperbolic in the sense of I. G. Petrovskii (it can be shown that, in the contrary case, the problem is, generally speaking, incorrectly posed).

Let us first consider the homogeneous system obtained from (1) for $F_i(x,t) \equiv 0$; denote it by (1′). The solution of problem (1′)–(2) satisfies the system of equations

$$u_i(x,t) = \sum_{j=1}^{n} s_{ij} f_j(x + \lambda_j t), \qquad i = 1, \dots, n; \tag{3}$$

$$\varphi_i(t) = \sum_{j=1}^{n} s_{ij} f_j(\lambda_{ij} t), \qquad i = 1, \dots, n, \tag{4}$$

where λ_i are the characteristic numbers of (1), $\lambda_{ij} = \lambda_j + \mu_i$, $i, j = 1, \dots, n$,

$$\|s_{ij}\| \, \|b_{ij}\| \, \|s_{ij}\|^{-1} = \begin{Vmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{Vmatrix}.$$

1. Let $x \in (0, \infty)$;

$$\rho_0 = \min_i \lim_{x \to 0} \frac{\ln|\varphi_i(x)|}{\ln x}; \qquad \rho_1 = \max_i \overline{\lim}_{x \to \infty} \frac{\ln|\varphi_i(x)|}{\ln x},$$

$\lambda_{ij} > 0$, $i, j = 1, \dots, n$ (geometrically the latter means that the straight lines l_i, $i = 1, \dots, n$, are not separated by the characteristics of (1)).

Then the solution (4), obtained by the usual operational method with the aid of the Mellin transform, is represented in the form

$$f_i(x) = \frac{1}{2\pi} \, \text{V. p.} \left\{ \int_{a_i - \infty}^{a_i + \infty} \sum \frac{\Delta_{ji}(w) \Phi_j^+(w)}{\Delta(w)} x^{-iw} \, dw + \int_{b_i - \infty}^{b_i + \infty} \sum \frac{\Delta_{ji}(w) \Phi_j^-(w)}{\Delta(w)} x^{-iw} \, dw \right\}, \tag{5}$$

where

$$\Delta(w) = \det \|s_{ij} \lambda_{ij}^{-iw}\|;$$

$\Delta_{ji}(w)$ is the algebraic complement of $s_{ij}\lambda_{ij}^{-iw}$ in $\Delta(w)$;

$$\Phi_j^+(w) = \int_1^\infty \varphi_j(t) t^{iw-1} \, dt; \qquad \Phi_j^-(w) = \int_0^1 \varphi_j(t) t^{iw-1} \, dt;$$

$a > \rho_1$, $b < \rho_0$ are such that

$$\Delta(ai+u)\neq 0,\qquad \Delta(bi+u)\neq 0\quad \text{for } u\in(-\infty,\infty).$$

From (5) we obtain

$$\min_i \lim_{x\to 0}\frac{\ln|f_i(x)|}{\ln x}\geq \rho_0;\qquad \max_i \overline{\lim}_{x\to\infty}\frac{\ln|f_i(x)|}{\ln x}\leq \rho_1. \tag{6}$$

Using Bochner' s results ([2]), which are carried over without difficulty, when $\lambda_{ij}>0$, to the system (4), one can prove that the solution (5) is unique in the class of functions satisfying (6) if and only if in the strip $\rho_0\leq \operatorname{Im} w\leq \rho_1$ there are no zeros of $\Delta(w)$. Hence Theorem 1 is obtained.

Theorem 1. If $\varphi_i(t),\ i=1,\dots,n$, are continuously differentiable and satisfy (6), and in the strip $\rho_0\leq \operatorname{Im} w\leq \rho_1$ there are no zeros of $\Delta(w)$, then in the class of functions for which

$$\max_i \overline{\lim}_{|x|+|t|\to\infty}\frac{\ln|u_i(x,t)|}{\ln(|x|+|t|)}\leq \rho_1;\qquad \min_{i,j}\lim_{s_i\to 0}\frac{\ln|u_j(x,t)|}{\ln s_i}\geq \rho_0,$$

where s_i is the distance of the point (x,t) from the i-th characteristic, problem $(1')$–(2) is posed correctly. If in the strip $\rho_0\leq \operatorname{Im} w\leq \rho_1$ there are zeros of $\Delta(w)$, then the problem is posed incorrectly.

Depending on ρ_0 and ρ_1, problem $(1')$–(2), when the conditions of Theorem 1 are fulfilled, may have solutions:

a) with a finite domain of dependence, when the solution at the point (x,t) depends on the values of $\varphi_i(t),\ i=1,\dots,n$, only in a finite neighborhood of $t=0$ (a problem of type I);

b) with an infinite domain of dependence, when the solution at the point (x,t) depends on the values of $\varphi_i(t),\ i=1,\dots,n$, outside some neighborhood of $t=0$ (a problem of type II);

c) with a mixed domain of dependence, when the solution at the point (x,t) depends on the values of $\varphi_i(t),\ i=1,\dots,n$, both in a neighborhood of $t=0$ and in a neighborhood of $t=\infty$ (a problem of type III).

It can be proved that the solution will have a finite domain of dependence if $\Delta(w)\neq 0$ for $\operatorname{Im} w\geq \rho_0$; an infinite one, if $\Delta(w)\neq 0$ for $\operatorname{Im} w\leq \rho_1$, and a mixed one in all remaining cases.

If $\varphi_i(t),\ i=1,\dots,n$, are given only for $0\leq t\leq T$, then problem $(1')$–(2) can be posed correctly only in the case when it belongs to type I.

Thus, the following theorem holds:

Theorem 2. If the continuously differentiable functions $\varphi_i(t)$, $i=1,\dots,n$, are given for $0\leq t\leq T$,

$$\min_i \lim_{t\to 0} \frac{\ln|\varphi_i(t)|}{\ln t} = \rho_0,$$

and in the half-plane $\rho_0 \leq \operatorname{Im}\omega$ there are no zeros of $\Delta(\omega)$, then problem (1′)–(2) is well posed in the class of functions for which

$$\min_{i,j} \lim_{s_i\to 0} \frac{\ln|u_j(x,t)|}{\ln s_i} \geq \rho_0.$$

If, however, for $\operatorname{Im}\omega \geq \rho_0$ there are zeros of $\Delta(\omega)$, then the problem is ill posed.

As an example, consider the system $du_1/dt = du_2/dx,\ du_2/dt = du_1/dx$. Its general solution is

$$u_1(x,t) = f_1(x+t) + f_2(x-t), \qquad u_2(x,t) = f_1(x+t) - f_2(x-t).$$

The functions $f_i(t)$ must satisfy the system

$$f_1((1+\mu)t) + f_2((1-\mu)t) = \varphi_1(t),$$

$$f_2((1+\nu)t) - f_2((1-\nu)t) = \varphi_2(t)$$

in order that $u_1(x,\mu x) = \varphi_1(x)$, $u_2(x,\nu x) = \varphi_2(x)$. Let, for definiteness, $|\nu| < 1$, $|\mu| < 1$, and $\mu < \nu$; then

$$f_1(t) = \varphi_1(t/(1+\mu)) - f_2(t(1-\mu)/(1+\mu)),$$

$$f_2(t) + f_2((1+\mu)(1-\nu)t : (1-\mu)(1+\nu))$$
$$= \varphi_1(t/(1-\mu)) + \varphi_2(t(1+\mu)/(1+\nu)(1-\mu)) \equiv \Phi(t),$$

or

$$f_2(t) + f_2(\alpha t) = \Phi(t),$$

where

$$\alpha = (1+\mu)(1-\nu)/(1-\mu)(1+\nu) < 1.$$

The roots of the equation $\Delta(\omega) = 1 + \alpha^{-i\omega} = 0$ in this case are the numbers

$$\omega_k = (2k+1)\pi/\ln\alpha$$

for arbitrary integers k. If $|\Phi(t)| < Ct^{\varepsilon}$ as $t \to 0$, $\varepsilon > 0$, then there exists a unique solution

$$f_2(t) = \Phi(t) - \Phi(\alpha t) + \cdots + (-1)^n \Phi(\alpha^n t) + \cdots,$$

for which

$$\lim_{t\to 0} \frac{\ln|f_2(t)|}{\ln t} > 0;$$

it depends on the values of $\Phi(\theta)$ at the points $\theta = t\alpha^r \le t$. If $|\Phi(t)| < Ct^{-\delta}$ as $t \to \infty$, $\delta > 0$, then there exists a unique solution

$$f_2(t) = \Phi(t/\alpha) - \Phi(t/\alpha^2) + \dots + (-1)^{n+1}\Phi(t/\alpha^n) + \dots$$

with the property

$$\lim_{t\to\infty} \frac{\ln|f_2(t)|}{\ln t} < 0.$$

It depends on the values of $\Phi(\theta)$ at the points $\theta = t/\alpha^r \ge t$. If, however, $|\Phi(t)| < Ct^{\varepsilon}$ as $t \to 0$ and $|\Phi(t)| < Ct^{-\delta}$ as $t \to \infty$, $\varepsilon > 0$, $\delta > 0$, then solutions of both types exist, each unique in its own class. The formulas relating $f_2(t)$ to $f_1(t)$ and $\{u_1(x,t), u_2(x,t)\}$ make it possible automatically to transfer the results found for $f_2(t)$ to the solution of the Goursat problem.

2. In the case of arbitrary signs of λ_{ij}, $i, j = 1, \dots, n$, we introduce the determinants

$$\Delta_1(\omega) = \det\left\|s_{ij}|\lambda_{ij}|^{-i\omega}\right\|$$

and

$$\Delta_2(\omega) = \det\left\|s_{ij}\operatorname{sign}\lambda_{ij}\,|\lambda_{ij}|^{-i\omega}\right\|;$$

the functions $\varphi_i(t)$ are prescribed for $-\infty < t < \infty$. Then Theorem 3 is valid.

Theorem 3. If the twice continuously differentiable $\varphi_i(t)$, $i = 1, \dots, n$, $-\infty < t < \infty$, satisfy the conditions

$$\min_i \lim_{|t|\to 0} \frac{\ln|\varphi_i(t)|}{\ln|t|} = \rho_0; \qquad \max_i \lim_{|t|\to\infty} \frac{\ln|\varphi_i(t)|}{\ln|t|} = \rho_1,$$

and in the strip $\rho_0 \le \operatorname{Im}\omega \le \rho_1$ there are no zeros of $\Delta_1(\omega)$ and $\Delta_2(\omega)$, then in the class of functions for which

$$\max_i \lim_{|x|+|t|\to\infty} \frac{\ln|u_i(x,t)|}{\ln(|x|+|t|)} \le \rho_1;$$

$$\min_{i,j} \lim_{s_i\to 0} \frac{\ln|u_j(x,t)|}{\ln|s_i|} \ge \rho_0,$$

problem (1′)–(2) is well posed. If in the strip $\rho_0 \le \operatorname{Im}\omega \le \rho_1$ there are zeros of $\Delta_1(\omega)$ or $\Delta_2(\omega)$, then the problem is ill posed.

Analogously to item 1, one can introduce three types of solutions; in particular, for the case where $\varphi_i(t)$, $i = 1, \dots, n$, are given for $-T \le t \le T$, one can obtain an analogue of Theorem 2.

3. If $F_i(x,t)$, $i = 1, \dots, n$, as $\rho = \sqrt{x^2+t^2} \to \infty$, grow no faster than a certain power ρ, then all the preceding results remain valid. Let $u_i^{(0)}(x,t)$, $i = 1, \dots, n$, be the solution of the Cauchy problem for (1) under the

conditions $u_i^{(0)}(x,0)=0$, and let $u_i^{(1)}(x,t)$ be the solution of the Goursat problem for system (1) under the conditions

$$u_i^{(1)}(x,t)\big|_{l_i}=\varphi_i(t)-u_i^{(0)}(x_0,t)\big|_{l_i},\qquad i=1,\dots,n;$$

then

$$u_i(x,t)=u_i^{(0)}(x,t)+u_i^{(1)}(x,t)$$

is a solution of problem (1)–(2). For the validity of all

of the preceding results for systems of type (1), in which $F_i,\ i=1,\dots,n$, depend on $x,t,u_1,\dots,u_n$, it is necessary to require that the restrictions on the growth of F_i be uniform with respect to $u_1,\dots,u_n$ throughout the whole domain of their variation.

4. Let us note, finally, that the main results of §§ 1-2 carry over without difficulty also to the case when $\varphi_i(t)$ and $u_i(x,t),\ i=1,\dots,n$, have discontinuities of a definite kind. Arbitrary discontinuities of the solutions, as the following example shows, cannot be allowed. Consider the problem: to find a solution of the system $\partial u_1/\partial t=\partial u_2/\partial x,\ \partial u_2/\partial t=\partial u_1/\partial x$ under the conditions $u_1|_{t=0}=0,\ u_2|_{t=\nu x}=0,\ |\nu|<1$. In the class of bounded solutions, which may have discontinuities along the characteristics of the system, there exists a nonzero solution of this problem:

$$u_1=\sin\frac{\pi\ln(x+t)/(x-t)}{\ln(1-\nu)/(1+\nu)}\sin\frac{\pi\ln(x^2-t^2)}{\ln(1-\nu)/(1+\nu)},$$

$$u_2=-\cos\frac{\pi\ln(x^2-t^2)}{\ln(1-\nu)/(1+\nu)}\cos\frac{\pi\ln(x+t)/(x-t)}{\ln(1-\nu)/(1+\nu)}.$$

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Note: Figure translations are in progress. See original paper for figures.