ON THE METHOD OF LINES FOR CERTAIN BOUNDARY-VALUE PROBLEMS FOR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

1957

SovietRxiv

Abstract

Full Text

MATHEMATICS

B. M. BUDYAK

ON THE METHOD OF LINES FOR CERTAIN BOUNDARY-VALUE PROBLEMS FOR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

(Presented by Academician A. A. Dorodnitsyn on 9 VIII 1956)

In the present note the convergence of the method of lines (^{1}) is proved, and the error incurred in solving by this method certain systems of partial differential equations is estimated.

1. Consider the linear spaces F and U of column matrices $\bar{f}(x,y)$ and $\bar{u}(x,y)$, containing N elements $f^{(i)}(x,y)$ and $u^{(i)}(x,y)$, continuous and sufficiently smooth in the rectangle $D: 0 \leq x \leq l',\ 0 \leq y \leq l''$, and also the linear spaces $F^{(n)}$ and $U^{(n)}$ of rectangular matrices $\bar{F}(y)$ and $\bar{U}(y)$, containing $(n+2)N$ elements $f_k^{(i)}(y)$ and $u_k^{(i)}(y)$, $i = 1, 2, \dots, N;\ k = 0, 1, \dots, n+1$ (i is the row number), continuous and sufficiently smooth on the interval $0 \leq y \leq l''$. We shall make these spaces normed by taking as the norm of a matrix the greatest of the least upper bounds of the absolute values of its elements*.

Divide the segment $0 \leq x \leq l'$ of the x-axis by the points $x_k = kh,\ (n+1)h = l'$, $k = 0, 1, \dots, n+1$, and from them draw ordinates up to their intersection with the upper side of the rectangle D; this system of rectilinear segments will be called the basic system of lines.

By $\Phi^{(n)}$ denote the mapping of F into $F^{(n)}$ and U into $U^{(n)}$, under which the elements of the image matrices $\bar{F}(y) = \Phi^{(n)}(\bar{f}(x,y))$ and $\bar{U}(y) = \Phi^{(n)}(\bar{u}(x,y))$ are equal to:

$$f_k^{(i)}(y) = f^{(i)}(x_k, y), \quad u_k^{(i)}(y) = u^{(i)}(x_k, y); \quad k = 0, 1, \dots, n+1; \quad i = 1, \dots, N. \tag{1}$$

By $\Lambda_{\Gamma'}(U)$ denote the linear manifold in U of functions $u(x,y)$ equal to zero on the part Γ' of the boundary Γ of the rectangle D, assuming that Γ' consists of one or several of its sides.

By $\Lambda_{\Gamma'}(U^{(n)}) = \Phi^{(n)}(\Lambda_{\Gamma'}(U))$ denote the corresponding linear manifold in $U^{(n)}$. We shall consider a linear differential operator L, mapping U into F, and a linear differential-difference operator L_n, mapping $U^{(n)}$ into $F^{(n)}$, which is an approximation to L. In doing so we shall suppose that, for the operator L_n considered on the linear manifold $\Lambda_{\Gamma'}(U^{(n)})$, there exists an inverse operator L_n^{-1}, defined on the linear manifold $\Lambda_{\Gamma'}(F^{(n)}) = L_n(\Lambda_{\Gamma'}(U^{(n)}))$.

Lemma 1. Let the equation

$$L(\bar{u}) = \bar{f}, \tag{2}$$

where $\bar{f} \in F$, have a solution $\bar{u}^* \in U$, taking prescribed values on Γ'. Let the approximate equation

$$L_n(\bar{U}) = \Phi^{(n)}(\bar{f}) \tag{3}$$

---

* In the same way we shall also define the norm of all other matrices occurring below.

has a solution $\overline{U}^*$ that assumes the same values at the corresponding points of Γ' as $\overline{u}^*$. Finally, let

$$\left\|L_n^{-1}\right\|_{\Lambda_{\Gamma'}(F^{(n)})} \leq C_n \tag{4}$$

and let the operator L_n approximate the operator L in the following sense:

$$\left\|\Phi^{(n)}(L(\overline{u})) - L_n(\Phi^{(n)}(\overline{u}))\right\|_{F^{(n)}} \leq \varepsilon_n(\overline{u}) \tag{5}$$

for every $u \in U$.

Then the solution $\overline{U}^*$ of the approximate equation (3) approximates the solution $\overline{u}^*$ of the exact equation (2) as follows:

$$\left\|\Phi^{(n)}(\overline{u}^*) - \overline{U}^*\right\|_{U^{(n)}} \leq C_n \varepsilon_n(\overline{u}^*). \tag{6}$$

The proof of this lemma can be obtained by a slight modification of the arguments in (^{2}) on pp. 107–108, or in (^{2}) on pp. 17–18.

We now turn to the consideration of concrete boundary-value problems. In Sec. 2 the estimate of the norm L_n^{-1} is carried out on the basis of an inductive pattern, while in Secs. 3 and 4 it is carried out with the aid of a differential-difference energy integral (^{3}).

2. Consider the Goursat problem

$$L(\overline{u}) \equiv \overline{u}_{xy} - \overline{A}(x,y)\,\overline{u} = \overline{f}(x,y), \qquad 0 \le x \le l', \quad 0 \le y \le l''; \tag{7}$$

$$\overline{u}\Big|_{x=0} = \overline{\varphi}(y), \quad 0 \le y \le l'', \quad \overline{u}\Big|_{y=0} = \overline{\psi}(x), \quad 0 \le x \le l'; \quad \overline{\varphi}(0) = \overline{\psi}(0), \tag{8}$$

where $\overline{A}(x,y)$ is a square matrix with N^2 elements $a^{(i,j)}(x,y)$; $\overline{\varphi}$, $\overline{\psi}$ are column matrices with N elements each. By Γ' we denote the part of the boundary of the rectangle D consisting of its left and lower sides.

Approximate values of the solution of this problem on the basic system of lines can be found as the values of the solution of the system of ordinary differential equations

$$\frac{\overline{u}_{k+1}(y) - \overline{u}_k(y)}{h} - \overline{A}(x_k,y)\,\overline{u}_k(y) = \overline{f}(x_k,y), \quad k = 0,1,\dots,n, \tag{9}$$

under the additional conditions:

$$\overline{u}_k(0) = \overline{\psi}(x_k), \quad \overline{u}_0(y) = \overline{\varphi}(y), \quad k = 0,1,\dots,n+1; \quad 0 \le y \le l''. \tag{10}$$

The left-hand sides of the system (9), (10) define the differential-difference operator L_n.

In case α), when $\overline{u}_{xy}(x,y)$ is continuous in D, one has

$$\left\|\Phi^{(n)}(L(\overline{u})) - L_n(\Phi^{(n)}(\overline{u}))\right\|_{F^{(n)}} \le \varepsilon_n(\overline{u}) \to 0 \quad \text{as } n \to +\infty. \tag{5_1^α}$$

In case β), when $\overline{u}_{xxy}(x,y)$ is continuous in D, one has

$$\left\|\Phi^{(n)}(L(\overline{u})) - L_n(\Phi^{(n)}(\overline{u}))\right\|_{F^{(n)}} \le \frac{l'\,\left\|\overline{u}_{xxy}\right\|}{2(n+1)}. \tag{5_1^β}$$

For the norm of the operator L_n^{-1} the following estimate holds:

$$\left\|L_n^{-1}\right\|_{\Lambda_{\Gamma'}(F^{(n)})} \le Q^{-1}(e^Q - 1)\,l'l'', \qquad Q = N\|\overline{A}\|\,l'l'', \tag{4_1}$$

whence, with the aid of relation (6) from Lemma 1, we obtain an estimate of the error $\|\Phi^{(n)}(\bar{u}^*) - \bar{U}^*\|$ in cases α) and β). Case α) for $\bar{u}^*$ holds if $\bar{\varphi}'(y)$ and $\bar{\psi}'(x)$ are continuous, respectively, on the intervals $0 \le y \le l''$ and $0 \le x \le l'$, and $\bar{f}(x,y)$ and $\bar{A}(x,y)$ are continuous in D. Case β) for $\bar{u}^*$ holds if, in addition, $\bar{f}_x$ and $\bar{A}_x$ are also continuous in D; moreover, the inequality is valid

$$\|\bar{u}^*_{xxy}\| \leq \frac{\|\bar{A}_x\|}{\|\bar{A}\|} \|\bar{f}^{(0)}\|(I_0(2\sqrt{\bar{Q}}) - 1) - l'' N\|\bar{A}\| \, \|\bar{f}^{(0)}\| I_0(2\sqrt{\bar{Q}}) + \|\bar{f}^{(0)}_x\|, \tag{11}$$

where

$$\bar{f}^{(0)}(x,y) = \bar{f}(x,y) + \bar{A}(x,y)[\bar{\psi}(x) + \bar{\varphi}(y) - \bar{\varphi}(0)]. \tag{12}$$

In case β) it is not difficult to obtain also a somewhat more precise estimate of the error:

$$\left|u^{(i)*}(x_k,y) - u^{(i)*}_k(y)\right| \leq \frac{e^{N\|A\|l''y} - 1}{N\|\bar{A}\|} \, \frac{l' \, \|\bar{u}^*_{xxy}\|}{2(n+1)}. \tag{13}$$

A similar estimate can also be obtained in the nonlinear case.

3. Consider the boundary-value problem

$$L(\bar{u}) \equiv \bar{R}(x)\bar{u}_{yy} - (\bar{S}(x)\bar{u}_x)_x + \bar{Q}(x)\bar{u} = \bar{f}(x,y), \quad 0 \leq x \leq l', \ 0 \leq y \leq l''; \tag{14}$$

$$\bar{u}(0,y) = \mu(y), \quad \bar{u}(l',y) = \nu(y), \quad 0 \leq y \leq l'', \tag{15}$$

$$\bar{u}(x,0) = \bar{\varphi}(x), \quad \bar{u}_y(x,0) = \bar{\psi}(x), \quad 0 \leq x \leq l', \tag{16}$$

where $\bar{R}(x), \bar{S}(x), \bar{Q}(x)$ are symmetric square matrices with N^2 elements, which are continuous and sufficiently smooth functions of x, $0 \leq x \leq l'$, with $\bar{R}(x)$ and $\bar{S}(x)$ positive definite, and $\bar{Q}(x)$ nonnegative definite on the interval $0 \leq x \leq l'$; $\bar{\mu}, \bar{\nu}, \bar{f}, \bar{u}$ are column matrices containing N elements each.

The approximate differential-difference boundary-value problem will be:

$$\bar{R}(x_k)\bar{u}''_k(y) - \frac{\bar{S}(x_k)[\bar{u}_{k+1}(y) - \bar{u}_k(y)] - \bar{S}(x_{k-1})[\bar{u}_k(y) - \bar{u}_{k-1}(y)]}{h^2} +$$

$$+\bar{Q}(x_k)\bar{u}_k(y) = \bar{f}(x_k), \quad k = 1, 2, \dots, n, \quad 0 \leq y \leq l''; \tag{17}$$

$$\bar{u}_0(y) = \bar{\mu}(y), \quad \bar{u}_{n+1}(y) = \bar{\nu}(y), \quad 0 \leq y \leq l'', \tag{18}$$

$$\bar{u}_k(0) = \bar{\varphi}(x_k), \quad \bar{u}'_k(0) = \bar{\psi}(x_k), \quad k = 0, 1, \dots, n+1. \tag{19}$$

Denote by Γ' the set of the lower horizontal and vertical sides of D. The left-hand sides of system (17) determine the differential-difference operator L_n. For the norm L_n^{-1} we obtain the estimate:

$$\|L_n^{-1}\|_{\Lambda_{\Gamma'}(F^{(n)})} \leq \frac{l'l''}{2}\sqrt{\frac{N}{\varkappa_{*R}\varkappa_{*S}}}, \tag{4_2}$$

where $\varkappa_{*R}$ and $\varkappa_{*S}$ are lower bounds of the least characteristic numbers of the matrices $\bar{R}(x)$ and $\bar{S}(x)$ on the interval $0 \leq x \leq l'$.

In case α), when $\bar{S}'(x)$ and $\bar{u}_{xx}(x,y)$ are continuous,

$$\|\Phi^{(n)}(L(\bar{u})) - L_n(\Phi^{(n)}(\bar{u}))\|_{F^{(n)}} \leq \varepsilon_n(\bar{u}) \to 0 \quad \text{as } n \to +\infty. \tag{5_2^α}$$

In case β), when $\bar{S}''(x)$ and $\bar{u}_{xxx}(x,y)$ are continuous,

$$\|\Phi^{(n)}(L(\bar{u})) - L_n(\Phi^{(n)}(\bar{u}))\|_{F^{(n)}} \leq \frac{N^{1/2}l'(\varkappa^*_{S''}\|\bar{u}_x\| + \varkappa^*_{S'}\|\bar{u}_{xx}\| + {}^2/_3\varkappa^*_S\|\bar{u}_{xxx}\|)}{2(n+1)}, \tag{5_2^β}$$

where $\varkappa^*_{S^r}$, $\varkappa^*_{S'}$, $\varkappa^*_S$ are the largest characteristic roots of the matrices obtained from $\overline{S}^r(x)$, $\overline{S}'(x)$, $\overline{S}(x)$ by replacing the elements by the maxima of their absolute values. Comparing (4_2), (5_2^α), (5_2^β), we, by virtue of relation (6) of Lemma 1, obtain an error estimate

$$\left\|\Phi^{(n)}(\overline{u}^*) - \overline{U}^*\right\|_{U^{(n)}}$$

in the cases α) and β). In the case β) one obtains also a more precise error estimate:

$$\left|u^{(i)*}(x_k, y) - u_k^{(i)*}(y)\right| \leq \frac{Nl'\left(\varkappa^*_{S''}\|\overline{u}_x\|\varkappa^*_{S'}\|\overline{u}_{xx}\| + \frac{2}{3}\varkappa^*_S\|\overline{u}_{xxx}\|\right) y\sqrt{x_k(l'-x_k)}}{2(n+1)\sqrt{\varkappa^*_R\varkappa^*_S}}. \tag{20}$$

If $\overline{R} = \overline{E}$, $\overline{\mu}(y) \equiv \overline{\nu}(y) \equiv 0$; $\overline{S}$ and $\overline{Q}$ are constant; φ, φ', φ'', φ''', ψ, ψ', ψ'' are continuous on the interval $0 \leq x \leq l'$ and vanish at its endpoints, while ψ''' and φ^{IV} are piecewise continuous and bounded in norm on it, and if, in addition, $\overline{f}$, $\overline{f}_x$, $\overline{f}_{xx}$ are continuous in D and vanish for $x = 0$ and $x = l'$, while $\overline{f}_{xxx}$ is piecewise continuous and bounded in norm in D, then for $\|\overline{u}_{xxx}\|$ one can obtain the estimate

$$\|\overline{u}_{xxx}\| \leq \frac{l'}{\sqrt{3}} N^{3/2}\Omega \left(\|\overline{\varphi}^{\text{IV}}\| + \frac{N\Omega}{\sqrt{\varkappa^*_S}}\left[\|\overline{\psi}''\| + l''\|\overline{f}_{xxx}\|\right]\right), \tag{21}$$

where

$$0 < \Omega < N\|\overline{T}\|\,\|\overline{T}^{-1}\|, \tag{22}$$

and $\overline{T}$ is a nonsingular matrix by means of which $\overline{S}$ and $\overline{Q}$ are transformed simultaneously to diagonal form; if $\overline{T}$ is orthogonal, then one may set $\Omega = 1$.

4. Let us consider the boundary-value problem for a generalized system of telegraph equations

$$\overline{L}\overline{i}_y + \overline{R}\overline{i} + \overline{v}_x = 0, \quad \overline{C}\overline{v}_y + \overline{G}\overline{v} + \overline{i}_x = 0, \qquad 0 \leq x \leq l', \quad 0 \leq y \leq l'', \tag{23}$$

$$\overline{v}(0,y) = \overline{\mu}(y), \quad \overline{v}(l',y) = \overline{\nu}(y), \quad 0 \leq y \leq l'', \tag{24}$$

$$\overline{v}(x,0) = \overline{\varphi}(x), \quad \overline{i}(x,0) = \overline{\psi}(x), \quad 0 \leq x \leq l', \tag{25}$$

where $\overline{L}$ and $\overline{C}$ are symmetric positive-definite constant square matrices with N^2 elements; $\overline{R}$ and $\overline{G}$ are likewise nonnegative-definite matrices; $\overline{\mu}(y)$, $\overline{\nu}(y)$, $\overline{\varphi}(x)$, $\overline{\psi}(x)$, $\overline{v}(x,y)$, $\overline{i}(x,y)$ are column matrices with N elements. Analogously to the preceding, we define a differential-difference approximate boundary-value problem with preservation of the derivatives with respect to y. For the norm L_n^{-1} we obtain the estimate

$$\|L_n^{-1}\|_{\Lambda\Gamma'(F^{(n)})} \le \frac{2l''\sqrt{nN}}{\varkappa}; \qquad \varkappa = \min(\varkappa_L^*, \varkappa_C^*). \tag{4_3}$$

In the case when $\overline{i}_{xx}$ and $\overline{v}_{xx}$ are continuous, there will be

$$\left\|\Phi^{(n)}(L(\overline{u})) - L_n(\Phi^{(n)}(\overline{u}))\right\|_{F(n)} \le \frac{l'\|\overline{u}_{xx}\|}{2n}, \tag{5_3}$$

where $\overline{u} = \binom{\overline{i}}{\overline{v}}$ is a column matrix with $2N$ elements.

By virtue of Lemma 1, from (4_3) and (5_3) we obtain the error estimate

$$\left\|\Phi^{(n)}(\overline{u}^*) - \overline{U}^*\right\|_{F(n)} \le \frac{l'l''}{\varkappa}\sqrt{\frac{N}{n}}\,\|\overline{u}^*_{xx}\|. \tag{26}$$

In conclusion, the author expresses his gratitude to A. A. Dorodnitsyn and V. A. Ditkin.

Moscow State University
named after M. V. Lomonosov

Received
15 VI 1956

CITED LITERATURE

1. S. G. Mikhlin, *Direct Methods in Mathematical Physics*, 1950.
2. L. V. Kantorovich, *Uspekhi Mat. Nauk*, **3**, no. 6 (28) (1948).
3. B. M. Budak, DAN, **109**, no. 1 (1956).
4. V. S. Ryaben' kii, A. F. Filippov, *On the Stability of Difference Equations*, 1956.

Note: Figure translations are in progress. See original paper for figures.