ON THE EQUICONVERGENCE OF FOURIER-CHEBYSHEV AND MACLAURIN EXPANSIONS OF ANALYTIC FUNCTIONS OF THE CLASS H_2

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Abstract

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MATHEMATICS

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ON THE EQUICONVERGENCE OF FOURIER-CHEBYSHEV AND MACLAURIN EXPANSIONS OF ANALYTIC FUNCTIONS OF THE CLASS H_2

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Let the polynomials $\{P_n(z)\}$ be orthonormal on the unit circle $z = e^{i\theta}$ with respect to the weight $p(\theta) \geq 0$, where $\lg p(\theta) \in L_1$; let the function $f(z)$ be regular in the domain $|z| < 1$, with $f(z) \in H_2$ and $f(z)/\pi(z) \in H_2$, where

$$\pi(z) = \exp\left\{-\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\lg p(\theta)\,d\theta\right\}, \quad |z| < 1; \tag{1}$$

denote by $s_n(f;z)$, $\sigma_n(f;z)$ the partial sums of the expansions of the function $f(z)$ in the Fourier-Chebyshev series with respect to the orthonormal polynomials $\{P_k(z)\}$ and in the Maclaurin series, i.e., with respect to the polynomials $\{z^k\}$:

$$s_n(f;z) = \sum_{k=0}^{n} c_{kP}k(z), \quad c_k = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{P_k(e^{i\theta})}p(\theta)\,d\theta;$$

$$\sigma_n(f;z) = \sum_{k=0}^{n}\gamma_k z^k, \quad \gamma_k = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ik\theta}\,d\theta. \tag{2}$$

Of all the theorems on convergence of Fourier-Chebyshev expansions, the theorem on the uniform equiconvergence of the Fourier-Chebyshev and Maclaurin expansions is of greatest interest, i.e., the conditions under which the limiting relation

$$\lim_{n\to\infty}\{s_n(f;e^{i\theta}) - \sigma_n(f;e^{i\theta})\} = 0 \tag{3}$$

holds uniformly within some interval $[\alpha,\beta] \subset [0,2\pi]$.

Theorem. *Let the weight $p(\theta)$ on the interval $[\alpha, \beta]$ be bounded below by a positive number and be continuous with modulus of continuity $\omega(\delta; p)$, satisfying the Dini-Lipschitz condition*

$$\omega(\delta; p) \leq C \left(\lg \frac{1}{\delta}\right)^{-\gamma}, \quad \gamma > 2; \tag{4}$$

let the function $f(z)$ have bounded radial boundary values at all points of the arc $[e^{i\alpha}, e^{i\beta}]$; then the condition $\lim_{n\to\infty}\{\varepsilon_n \lg n\} = 0$, where

$$|P_n^*(e^{i\theta}) - \pi(e^{i\theta})| \leq \varepsilon_n, \quad P_n^*(z) = z^n \overline{P_n}\left(\frac{1}{z}\right), \quad \alpha + \eta \leq \theta \leq \beta - \eta, \tag{5}$$

is sufficient for the uniform convergence of (3) on the interval $[\alpha + \eta', \beta - \eta']$, $\eta' > \eta$.

* For the proof of the theorem it suffices to apply the argument of G. Szegő ((1), §§ 13.3, 13.7) with the corresponding generalizations and modifications.

Thus, for uniform equiconvergence under the indicated conditions it is sufficient that there exist an asymptotic formula (5) with error $\varepsilon_n = o\left(\frac{1}{\lg n}\right)$.

	Conditions imposed on the weight $p(\theta)$on the interval $[0, 2\pi]$	Conditions imposed on the weight $p(\theta)$on the interval $[\alpha, \beta]$	Estimate for ε_n
I	$p(\theta) \leq M, \frac{1}{p(\theta)} \in L_1$	$\omega(\delta; p) \leq C\left(\lg \frac{1}{\delta}\right)^{-\gamma}$,$\gamma > 2$	$C_1\omega_4\left(\frac{1}{n}; \lg p\right) + C_2\omega_2\left(\frac{1}{n}; \frac{1}{\sqrt{p}}\right) + + C_3(\lg n)^{2-\gamma}$ or$C_4\sqrt{\omega_1\left(\frac{1}{n}; \frac{1}{p}\right)} + C_3(\lg n)^{2-\gamma}$
II	$p(\theta) \leq M, \frac{1}{p(\theta)} \in L_2$	ω the same,$\gamma > 1$	$C_1\omega_4\left(\frac{1}{n}; \lg p\right) + C_2\omega_2\left(\frac{1}{n}; \frac{1}{p}\right) + + C_3(\lg n)^{1-\gamma}$
III	$0 < m \leq p(\theta)$	ω the same,$\gamma > 2$	$C_1\omega_4\left(\frac{1}{n}; \lg p\right) + C_2\omega_2\left(\frac{1}{n}; \sqrt{p}\right) + + C_3(\lg n)^{2-\gamma}$ or$C_4\sqrt{\omega_1\left(\frac{1}{n}; p\right)} + C_3(\lg n)^{2-\gamma}$

	Conditions imposed on the weight $p(\theta)$on the interval $[0, 2\pi]$	Conditions imposed on the weight $p(\theta)$on the interval $[\alpha, \beta]$	Estimate for ε_n
IV	$p(\theta) \in L_r, \dfrac{1}{p(\theta)} \in L_{r'}, \dfrac{1}{r} + \dfrac{1}{r'} = 1,\ r > 1$	ω the same,$\gamma > 2$	$C_1\omega_{2r}\left(\dfrac{1}{n}; \sqrt{p}\right) + C_2\omega_4\left(\dfrac{1}{n}; \lg p\right) + + C_3(\lg n)^{2-\gamma}$ or$C_4\sqrt{\omega_r\left(\dfrac{1}{n}; p\right)} + C_3(\lg n)^{2-\gamma}$
V	$\lim\limits_{\delta\to 0} I(\delta) = 0$	ω the same,$\gamma > 2$	$C_1\sqrt{I\left(\dfrac{1}{n}\right)} + C_2(\lg n)^{2-\gamma}$

The table presents the 5 conditions found by us, each of which is sufficient for the existence of the asymptotic formula (5), provided only that the weight is bounded below by a positive number and is continuous on the interval $[\alpha, \beta]$; by $\omega_r(\delta; \varphi)$ is denoted the integral modulus of continuity of the function $\varphi(\theta)$ in the metric of the space $L_r,\ r > 1$; by $I(\delta)$ is denoted the quantity*

$$I(\delta) = \sup_{|h|\le\delta}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{|p(\theta+h) - p(\theta)|}{p(\theta)}\,d\theta\right\}; \tag{6}$$

the last column gives estimates of the error ε_n.

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CITED LITERATURE

1. G. Szegő, *Orthogonal Polynomials*, N. Y., 1939.

2. G. Freud, *Acta Math. Acad. Sci. Hung.*, 5, No. 3—4 (1954).

* It could be called the weighted integral modulus of continuity of the weight $p(\theta)$ in the metric of the space L_1; it was first introduced by G. Freud (2).

Note: Figure translations are in progress. See original paper for figures.