



Soviet-era science, translated into English

MATHEMATICS

B. V. SHABAT

1957

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-195701.81412>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

B. V. SHABAT

EXAMPLES OF SOLVING THE DIRICHLET PROBLEM FOR EQUATIONS OF MIXED TYPE

(Presented by Academician M. A. Lavrent'ev, 28 VII 1956)

In the study of established two-dimensional vortex-free flows of an ideal gas in nozzles, two ways of passing through the speed of sound are considered: a) the speed of sound is reached in some cross-section of the nozzle, close to the minimal one, so that the transition line extends from one wall to the other; b) supersonic zones adjoin the walls of the nozzle near the minimal cross-section. In the hodograph plane, these two ways of transition correspond to two boundary-value problems for linear second-order partial differential equations of mixed type, which in a simplified formulation reduce, respectively, to the Tricomi and Dirichlet problems.

While an extensive literature is devoted to the first of these problems, the second, as far as we know, has not been considered in the literature. In the present note we give several examples relating to the formulation of the second problem, for simplicity restricting ourselves to the model equation of mixed type ("Laplace –string"), proposed by M. A. Lavrent'ev ⁽¹⁾ and successfully studied by A. V. Bitsadze ⁽²⁾ and others.

1°. Let us begin with the solution of the Dirichlet problem for the simplest equation of hyperbolic type

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0. \quad (1)$$

Let the domain D be a crescent bounded by the segment $[0, 1]$ of the x -axis and the curve $L: y = -y(x)$, where $y(x)$ is a twice differentiable function satisfying the conditions

$$y(x) > 0 \quad \text{for} \quad 0 < x < 1; \quad y(0) = y(1) = 0; \quad |y'(x)| \leq 1, \quad (2)$$

with equality attained at isolated points. Let a function $F_1(x)$ be prescribed on the segment, and a function $F_2(x)$ on L , with F_1 and F_2 twice differentiable and $F_1(0) = F_2(0)$, $F_1(1) = F_2(1)$. It is required to find a twice differentiable

function $u(x, y)$, satisfying equation (1) in D and taking the prescribed values on the boundary.

Theorem 1. *Under the adopted conditions, the Dirichlet problem for equation (1) has a continuum of solutions which, generally speaking, have no limit as $(x, y) \rightarrow (0, 0)$. The condition of existence of such a limit selects the unique solution of the problem.*

For the proof, consider the general solution of equation (1)

$$u = f(x + y) + f_1(x - y), \quad (3)$$

where f and f_1 are arbitrary twice differentiable functions. Substitution of the boundary values leads to the functional equation

$$f[\lambda(x)] - f(x) = F(x), \quad 0 < x < 1, \quad (4)$$

where the strictly increasing function $\lambda(x)$ ($\lambda(x) < x$ for $0 < x < 1$, $\lambda(0) = 0$, $\lambda(1) = 1$) is determined from the equations

$$t + y(t) = x, \quad t - y(t) = \lambda(x) \quad (5)$$

and

$$F(x) = F_1(x) - F_2 \left[\frac{x + \lambda(x)}{2} \right]$$

is a known function, $F(0) = F(1) = 0$.

The functional equation (4) has a continual set of solutions: one may arbitrarily specify the function $f(x)$ on some interval $[\lambda_1, \lambda_0]$, where $\lambda_1 = \lambda(\lambda_0)$, and then, according to (4), continue this function to the whole interval $[0, 1]$. If, in addition, the prescribed function $f(x)$ is made to satisfy the corresponding conditions at the endpoints of the interval $[\lambda_1, \lambda_0]$, then the function obtained by continuation will be twice differentiable; however, generally speaking, it has no limit as $x \rightarrow 0$. Thus there exists a continual set of solutions of the problem under consideration.

Under the additional assumption that $\lim_{x \rightarrow 0} f(x)$ exists, the solution of equation (4) is determined in a unique manner, if constant terms, immaterial for our purposes, are disregarded. Indeed, iterating relation (4), we find

$$f(x) = f[\lambda^n(x)] - \sum_{k=0}^{n-1} F[\lambda^k(x)]$$

(we put $\lambda^0(x) = x$, $\lambda^k(x) = \lambda[\lambda^{k-1}(x)]$); hence it is clear that if there exists a solution of equation (4) having $\lim_{x \rightarrow 0} f(x) = A$, then the sum on the right-hand side also has a limit and

$$f(x) = A - \sum_{k=0}^{\infty} F[\lambda^k(x)]$$

is completely determined. The theorem is proved.

One can also indicate sufficient conditions imposed on the functions $\lambda(x)$ and $F(x)$ under which there exists a solution of the problem having a limit as $(x, y) \rightarrow (0, 0)$. The simplest of such conditions is the possibility of representing these functions near $x = 0$ in the form $\lambda(x) = \alpha x + o(x)$, $0 \leq \alpha < 1$, and $F(x) = ax + o(x)$, where $o(x)$ is a quantity of higher order of smallness relative to x .

2°. Let us now consider the case of solving the Dirichlet problem for an equation of mixed type, which may be regarded as limiting for the generalized Tricomi problem. For simplicity suppose that the domain D is the upper semicircle D_1 with diameter $[0, 1]$, supplemented by a little lune D_2 , whose boundary L satisfies conditions (2). In this domain consider the equation

$$\frac{\partial^2 u}{\partial x^2} + \text{sign } y \frac{\partial^2 u}{\partial y^2} = 0 \quad (6)$$

of elliptic type in D_1 and hyperbolic type in D_2 ($\text{sign } y = y/|y|$ —the sign of y). By its solution we shall mean a function $u(x, y)$, continuous in D together with its first-order partial derivatives and, for $y \neq 0$, having second partial derivatives satisfying (6).

Theorem 2. *If the curve L satisfies, in addition to condition (1), also the condition*

$$y'(x) \leq \frac{y(x)}{x - x^2 + y^2(x)}, \quad 0 \leq x \leq 1, \quad (7)$$

where equality is attained only at isolated points, then in the class of functions continuous in \bar{D} , the Dirichlet problem for equation (6) cannot have more than one solution.

The proof differs only inessentially from the proof of the corresponding theorem of A. V. Bitsadze ((²), §13).

Remark. Condition (7) is satisfied, in particular, by all parabolas $y = ax(1-x)$ for $a < 1$.

Theorem 3. *Under the conditions of Theorem 2 and under the additional condition*

$$|y'(x)| \leq q < 1 \quad (8)$$

the Dirichlet problem for the mixed-type equation (6) is solvable.

In the domain D_2 we represent the solution in the form (3) and, repeating with slight changes the reasoning of A. V. Bitsadze [2], arrive at the equation

$$f(x) + f[\lambda^2(x)] - \int_0^1 k(x, t) f[\lambda(t)] dt = g(x), \quad 0 \leq x \leq 1, \quad (9)$$

where

$$k(x, t) = \frac{1}{\pi^2} \sqrt{\frac{x(1-x)}{t(1-t)}} \int_0^1 \sqrt{\frac{\lambda(\tau)[1-\lambda(\tau)]}{\tau(1-\tau)}} \left(\frac{1}{\tau-x} - \frac{1}{\tau+x-2\tau x} \right) \times \\ \times \left(\frac{1}{t-\lambda(\tau)} - \frac{1}{t+\lambda(\tau)-2t\lambda(\tau)} \right) d\tau$$

and $g(x)$ is a known function ([2], p. 49); $k(0, t) \equiv k(1, t) \equiv 0$, $g(0) = g(1) = 0$.

From condition (8) it is seen that the series

$$K(x, t) = \sum_{n=0}^{\infty} (-1)^n k[\lambda^{2n}(x), t], \quad G(x) = \sum_{n=0}^{\infty} (-1)^n g[\lambda^{2n}(x)]$$

converge on any segment $[0, x_0]$, $x_0 > 1$; since $\lambda(1) = 1$ and $k(1, t) = g(1) = 0$, they also converge for all x in the segment $[0, 1]$. Iterating (9) and assuming, without loss of generality, that $f(0) = 0$, we obtain for $0 \leq x \leq 1$

$$f(x) - \int_0^1 K(x, t) f[\lambda(t)] dt = G(x)$$

(the legality of the passage to the limit under the integral sign is easily verified).

Replacing here x by $\lambda(x)$ and denoting $f[\lambda(x)] = h(x)$, $G[\lambda(x)] = H(x)$, $K[\lambda(x), t] = L(x, t)$, we obtain an equation of Fredholm type

$$h(x) - \int_0^1 L(x, t) h(t) dt = H(x).$$

It follows from Theorem 2 that it is solvable, and this proves the theorem.

3°. As the next example, let us consider the solution of the Dirichlet problem for an equation of elliptic type in the disk $\rho < 1$ and of hyperbolic type in the annulus $1 < \rho < h$:

$$\rho^2 \frac{\partial^2 u}{\partial \rho^2} + \rho \frac{\partial u}{\partial \rho} + \text{sign}(1 - \rho) \frac{\partial^2 u}{\partial t^2} = 0 \quad (10)$$

(ρ and t are polar coordinates); the boundary values are prescribed on the circle $\rho = h$.

Theorem 4. *There exists an everywhere dense set of values $h > 1$ for which the Dirichlet problem for equation (10) has a unique solution for arbitrary (twice differentiable) boundary values, and also an everywhere dense set of values h for which this problem either is not solvable or has a continuum of solutions.*

For the proof, note that in the annulus $1 < \rho < h$ the solution of equation (10) is represented in the form

$$u = f(\ln \rho + t) + f_1(\ln \rho - t).$$

Let $\varphi(t)$ denote the values of this solution on the circle $\rho = 1$, and from these values construct a function harmonic in the disk $\rho < 1$. Writing the condition of continuity of $\partial u / \partial \rho$ on the circle $\rho = 1$, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi'(\tau) \text{ctg} \frac{\tau - t}{2} d\tau = 2f'(t) - \varphi'(t).$$

This singular integral equation is easily inverted:

$$\varphi'(t) = f'(t) - \frac{1}{2\pi} \int_0^{2\pi} f'(\tau) \text{ctg} \frac{\tau - t}{2} d\tau. \quad (11)$$

Substitution of the boundary values on the circle $\rho = h$ leads to the relation $f'(t+a) - f'(t-a) + \varphi'(t-a) = F'(t)$, where $a = \ln h$ and $F(t)$ are the values prescribed on $\rho = h$. Replacing here t by $t+a$ and substituting, in place of $\varphi'(t)$, its expression from (11), we obtain

$$f'(t+2a) - \frac{1}{2\pi} \int_0^{2\pi} f'(\tau) \text{ctg} \frac{\tau - t}{2} d\tau = F'(t+a).$$

By methods usual in the theory of integral equations (see, for example, (3)), we find

$$\frac{1}{2\pi} \int_0^{2\pi} f'(\tau + 2a) \operatorname{ctg} \frac{\tau - t}{2} d\tau + f'(t) = \frac{1}{2\pi} \int_0^{2\pi} F'(\tau + a) \operatorname{ctg} \frac{\tau - t}{2} d\tau.$$

Replace in the integral on the left the variable τ by $\tau - 2a$ and add this equation to the preceding one, in which t is replaced by $t + 2a$; we obtain

$$f'(t + 4a) + f'(t) = G(t), \quad (12)$$

where

$$G(t) = F'(t + 3a) + \frac{1}{2\pi} \int_0^{2\pi} F'(\tau + a) + \frac{\tau - t}{2}$$

is a known function.

The functional equation (12) is easily solved by means of Fourier series: if we put

$$G(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt} \quad \text{and} \quad f'(t) = \sum_{k=-\infty}^{\infty} \gamma_k e^{ikt},$$

then, for determining the unknown coefficients γ_k , we shall have the system of equations

$$2\gamma_k \cos 2ak = c_k e^{-2aik}. \quad (13)$$

If

$$a \neq \frac{2m + 1}{2n} \frac{\pi}{2},$$

where m and n are natural numbers, then the system (13) is solvable for arbitrary c_k ; moreover, if a/π is an algebraic number, it is easy to prove the convergence of the series for f' together with the series for G . If, however,

$$a = \frac{2m + 1}{2n} \frac{\pi}{2},$$

then the system (13), generally speaking, is not solvable. It turns out to be solvable only under the additional condition that the corresponding coefficients c_k vanish; but in this case it is solved non-uniquely—up to any function

whose Fourier expansion contains harmonics only of those numbers k for which $\cos 2ak = 0$. The theorem is proved.

4°. Consider again the equation of mixed type (6), but now in the half-plane $y > -h$ ($h > 0$); we prescribe the boundary value on the line $y = -h$. We suppose that the prescribed function is twice differentiable and, together with its first derivative, satisfies a Hölder condition at infinity with exponent $\alpha < 1$.

Theorem 5. *The formulated Dirichlet problem for equation (6) has, for any h , a continuous set of solutions. The additional assumption on the existence of the limit of the derivative of the solution as $x \rightarrow -\infty$ selects a unique solution of the problem.*

The proof is analogous to the preceding one.

Received
9 VI 1956

REFERENCES

- ¹ M. A. Lavrent'ev, A. V. Bitsadze, DAN, 70, No. 3, 373 (1950).
- ² A. V. Bitsadze, Tr. Inst. im. V. A. Steklova, Academy of Sciences of the USSR, 41 (1953).
- ³ S. G. Mikhlin, *Integral Equations*, Moscow, 1949.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.