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1957

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Abstract

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MATHEMATICS

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ON THE BEHAVIOR OF CYCLES UNDER CONTINUOUS MAPPINGS OF COMPACTA

(Presented by Academician P. S. Aleksandrov on 13 II 1957)

§ 1. Let M^n and M_1^n be two n-dimensional closed orientable manifolds. If f is a continuous mapping of M^n into M_1^n of degree zero, then z^n, the fundamental n-dimensional cycle of the manifold M^n, is mapped to zero. In particular, this holds for a mapping of M^n into n-dimensional Euclidean space.

P. S. Aleksandrov posed the problem: to prove that for every k, $0 \le k \le n-1$, there exists in M^n an essential k-dimensional cycle z^k that is mapped to zero. A positive answer to the question posed, in a more general formulation, is given in this note. The group of coefficients is taken to be the group of rational numbers.

An **essential cycle** is a cycle not ~ 0 on some carrier of its own. If a cycle is essential, then its **essential carrier** is such a carrier on which the cycle is not ~ 0. We shall say that a k-dimensional essential cycle z^k is mapped to zero **by means of** f (written $f(z^k) = 0$) if some essential carrier Φ of the cycle z^k is mapped either into a set containing no nonzero k-dimensional cycles, or into a k-dimensional orientable manifold M^k, and the cycle $f(z^k) \sim 0$ in M^k. In what follows, by the words carrier of an essential cycle we shall always mean an essential carrier.

§ 2. In Theorems 1, 2, 3 the cycle z^k, which is mapped to zero, $0 \le k \le n-1$, is a cycle arising as the result of a "cutting" of a $(k+1)$-dimensional cycle z^{k+1}. This means the following. There is a theorem, proved by P. S. Aleksandrov and called by him the "Fragma–Brouwer theorem."

Let a compactum B, lying in Euclidean space R^m and being an essential carrier of the cycle z^{k+1}, be represented as the sum of two compacta: $B = B_1 \cup B_2$. If B_1 and B_2 are such that a cycle y^q, linked with z^{k+1} and lying in $R^m \setminus B$, is homologous to zero in $R^m \setminus B_1$ and in $R^m \setminus B_2$, then in $B_1 \cap B_2$ there lies a k-dimensional cycle z^k, not homologous to zero in $B_1 \cap B_2$. This cycle $z^k \sim 0$ in B_1 and in B_2 and is the boundary of the chain formed by all simplices of the cycle z^{k+1} that have at least one vertex in B_1. Of such a cycle z^k we shall say that it is formed as the result of "cutting" the cycle z^{k+1}.

Theorem 1. *Let a compactum F, lying in Euclidean space R^m, be an essential carrier of the n-dimensional cycle z^n. Let f be such a continuous mapping of the compactum F into an n-dimensional closed orientable manifold M^n that the*

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cycle z^n is mapped to zero. Then for every k, $0 \leq k \leq n-1$, in F there exists a k-dimensional

nontrivial cycle z^k, homologous to zero in F, which is mapped to zero.

If F is taken to be an n-dimensional manifold, then one obtains the answer to the problem posed in § 1.

The proof of Theorem 1 is obtained by successive application of Theorem 2.

Theorem 2. *Let a compact set F, lying in R^m, be an essential carrier of an n-dimensional cycle z^n, and let f be such a continuous mapping of F into an n-dimensional closed orientable manifold M^n that $f(z^n) = 0$. Then in F there exists an $(n-1)$-dimensional cycle z^{n-1}, homologous to zero in F, and such an essential carrier F_1 of the cycle z^{n-1} that the set $f(F_1)$ lies in a polyhedron S^{n-1}, homeomorphic to an $(n-1)$-dimensional sphere, and the cycle $f(z^{n-1}) \sim 0$ in S^{n-1}, i.e. $f(z^{n-1}) = 0$.*

Proof. Let y^q be a polyhedral cycle in $R^m \setminus F$, linked with z^n, $q + n = m - 1$. Number all n-dimensional simplexes of M^n in such a sequence $T_1, T_2, \dots, T_N$ that, for any $k > 1$, the simplex T_k adjoins at least one of the simplexes $T_1, T_2, \dots, T_{k-1}$ along an $(n-1)$-dimensional face.

Suppose first that none of the sets $f^{-1}(\overline{T}_j)$, $j = 1, 2, \dots, N$, contains a set linked with y^q (1), i.e.

$$y^q \sim 0 \text{ in } R^m \setminus f^{-1}(\overline{T}_j), \qquad j = 1, 2, \dots, N. \tag{α}$$

Denote by s such a natural number that

$$y^q \nsim 0 \text{ in } R^m \setminus f^{-1}\left(\bigcup_{j=s}^{N} \overline{T}_j\right),$$

but

$$y^q \sim 0 \text{ in } R^m \setminus f^{-1}\left(\bigcup_{j=s+1}^{N} \overline{T}_j\right). \tag{1}$$

Such an s will be found, since, according to (α), we have $y^q \sim 0$ in $R^m \setminus f^{-1}(\overline{T}_N)$ and $y^q \nsim 0$ in

$$R^m \setminus f^{-1}\left(\bigcup_{j=1}^{N} \overline{T}_j\right) = R^m \setminus F,$$

and, consequently, among the numbers $1, 2, \dots, N-1$ there will be one satisfying condition (1).

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Case $s = 1$. Put

$$B_1 = \overline{T}_1, \qquad B_2 = \bigcup_{j=2}^{N} \overline{T}_j, \qquad A_1 = f^{-1}(B_1),$$

$$A_2 = f^{-1}(B_2).$$

According to (α), $y^q \sim 0$ in $R^m \setminus A_1$ and, by virtue of (1), $y^q \sim 0$ in $R^m \setminus A_2$, but $y^q \nsim 0$ in $R^m \setminus (A_1 \cup A_2) = R^m \setminus F$.

According to the theorem of Fragmaen-Brouwer[1], the set $A = A_1 \cap A_2$ contains such an $(n-1)$-dimensional cycle z^{n-1} that $z^{n-1} \nsim 0$ on A and z^{n-1} bounds a chain x_1 consisting of all simplexes of the cycle z^n having at least one vertex in A_1. We have $f(A) \subset B_1 \cap B_2$, i.e. $f(A)$ is contained in the boundary of the simplex T_1, which we denote by S^{n-1}. Therefore

$$f(z^{n-1}) = cz_1^{n-1},$$

where z_1^{n-1} is the fundamental cycle on S^{n-1}.

The chain $f(x_1) = f(z^n) \cap T_1$ is the part of the cycle $f(z^n)$ lying on T_1; since $f(z^n) = 0$, $f(x_1)$ covers the simplex T_1 with degree zero. Further, since

$$\Delta f(x_1) = f(\Delta x_1) = f(z^{n-1}) = cz_1^{n-1},$$

it follows that $c = 0$. The cycle z^{n-1} and $F_1 = A$ are the desired ones in this case.

Case $s > 1$. Put

$$B_1 = \overline{T}_s, \qquad B_2 = \bigcup_{j=s+1}^{N} \overline{T}_j, \qquad A_1 = f^{-1}(B_1), \qquad A_2 = f^{-1}(B_2).$$

As in the case $s = 1$, we obtain that $A_1 \cap A_2$ is an essential carrier of an $(n-1)$-dimensional cycle z^{n-1}. The set $B_1 \cap B_2$ lies in the part

boundary of the simplex T_s, which does not contain the interiors of those $(n-1)$-dimensional faces along which T_s is adjacent to the simplexes $T_1, T_2, \dots, T_{s-1}$. Therefore $B_1 \cap B_2$ contains no nonzero $(n-1)$-dimensional cycles, and $f(z^{n-1}) = 0$, since its carrier $F_1 = A_1 \cap A_2$ is mapped into $B_1 \cap B_2$. It remains to consider the case when (α) is not fulfilled. We have, for some j_0, $1 \leq j_0 \leq N$, $y^q \sim 0$ in $R^m \setminus f^{-1}(\overline{T}_{j_0})$. Represent T_{j_0} as an n-dimensional cube J^n. The set $f^{-1}(T_{j_0})$ contains a set B linked with y^q. If the set $f(B)$ is a single point, then, since B contains a compactum F_1 that is an essential carrier of an $(n-1)$-dimensional cycle z^{n-1}, homologous to zero in B (([1]), 1.90), the cycle z^{n-1} is the one sought –its carrier F_1 is mapped to the point a. In the contrary case, take two points p_1 and p_2 in B such that $f(p_1) \neq f(p_2)$. Regard the cube J^n as situated in n-dimensional Euclidean space and separate the points $f(p_1)$ and $f(p_2)$ by an

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$(n-1)$-dimensional hyperplane E^{n-1}; denote the closed half-spaces containing the points $f(p_1)$ and $f(p_2)$, respectively, by E_1^n and E_2^n. The sets

$$B_1 = [f^{-1}(E_1^n \cap J^n)] \cap B$$

and

$$B_2 = [f^{-1}(E_2^n \cap J^n)] \cap B$$

are proper subsets of B; hence $y^q \sim 0$ in $R^m \setminus B_1$ and in $R^m \setminus B_2$. The set $B = B_1 \cup B_2$; by the theorem of Phragmén–Brouwer, $B_1 \cap B_2 = f^{-1}(E^{n-1} \cap J^n)$ is the carrier of a cycle z^{n-1}, homologous to zero in B_1 and in B_2. Since $E^{n-1} \cap J^n$ contains no $(n-1)$-dimensional cycles, the cycle z^{n-1} is the one sought. The theorem is proved.

From the proof just given of Theorem 1 one can obtain Theorem 3, whose direct proof is considerably simpler.

Theorem 3. *Let a compactum F, lying in Euclidean space R^m, be an essential carrier of an n-dimensional cycle z^n. Let f be a continuous mapping of F into n-dimensional Euclidean space R^n, or let f be such a continuous mapping of F into an n-dimensional orientable manifold M^n that $f(F)$ does not contain some point a of the manifold M^n. Then for every k, $0 \leq k \leq n-1$, there exists in R^n or in M^n a set homeomorphic to k-dimensional Euclidean space, whose preimage is an essential carrier of a k-dimensional cycle z^k, homologous to zero in F, i.e. $f(z^k) = 0$.*

§ 3. Under a mapping of an n-dimensional orientable manifold or an n-dimensional cycle into n-dimensional Euclidean space, the following theorems hold.

Theorem 4. *Let f be a continuous mapping of an n-dimensional closed orientable manifold M^n into Euclidean space R^n. Let $q < n-1$, and let ζ^q be a q-dimensional cycle of the manifold M^n, homologous to zero in M^n, such that the cycle $f(\zeta^q) \sim 0$ on $f(\Phi)$, where Φ is an essential carrier of ζ^q. Then in R^n there exists a polyhedron Q, containing no nonzero p-dimensional cycles, $p+q = n-1$, whose preimage is an essential carrier of a p-dimensional cycle z^p, homologous to zero in M^n. Moreover, either z^p is linked with a q-dimensional cycle ζ_1^q obtained from ζ^q by an ε-modification, or the polyhedron Q is the sum of a finite number of simple polygonal lines having no common points.*

Theorem 5. *Let f be a continuous mapping of an n-dimensional polyhedral cycle M^n into R^n, and let $q < n-1$ and ζ^q be a q-dimensional cycle in M^n, lying entirely in one of the n-dimensional simplexes of some triangulation of M^n. If in M^n there exists such an essential carrier Φ of the cycle ζ^q that the cycle $f(\zeta^q) \sim 0$ on $f(\Phi)$, then in R^n there exi*

there exists a polyhedron Q, containing no nonzero p-dimensional cycles, $p + q = n-1$, whose preimage is an essential carrier of a p-dimensional cycle z^p, homologous to zero in M^n. In this case either $z^p \sim 0$ in $M^n \setminus \Phi_1$, where Φ_1 is the carrier of the q-dimensional cycle z_1^q obtained from ζ^q by means of

SovietRxiv

an ε-modification, or the polyhedron Q is a sum of a finite number of simple polygonal lines having no points in common.

In conclusion I express my gratitude to P. S. Aleksandrov and K. A. Sitnikov for the great assistance rendered in solving the problem.

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Received
13 II 1957

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1. P. S. Alexandroff, *Dimensions Theorie*, 1932, pp. 106, 161.

Note: Figure translations are in progress. See original paper for figures.

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