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Abstract

Full Text

MATHEMATICS

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ON THE PROPERTIES OF SOLUTIONS OF CERTAIN BOUNDARY-VALUE PROBLEMS FOR EQUATIONS OF PARABOLIC TYPE

(Presented by Academician S. L. Sobolev on 18 VI 1957)

In the present paper we prove the uniqueness of certain boundary-value problems for partial differential equations of parabolic type and the continuous dependence of solutions on the coefficients of the equation, on the right-hand sides, and on the initial and boundary conditions. The method developed by Hopf ^(1,2) and O. A. Oleinik ⁽³⁾ is used.

Let G be a domain in the n -dimensional Euclidean space E_n , the points of which we denote by $X = (x_1, \dots, x_n)$; for brevity we shall write $|X|^2 = x_1^2 + \dots + x_n^2$, $f(X) = f(x_1, \dots, x_n)$, $u(X, t) = u(x_1, \dots, x_n, t)$, etc. Denote the cylinder $G \times (0, T)$ by the symbol V , its lower base by the symbol Z , and its lateral surface ($0 < t < T$) by the symbol S . Put $V_t = G \times (0, t)$ for $0 < t < T$. Let

$$L(u) = \sum_{i,j=1}^n a_{ij} u_{x_i x_j} + \sum_{i=1}^n b_i u_{x_i} - u_t + cu$$

be a linear differential operator whose coefficients are continuous and bounded functions on $V + S + Z$, with $c \leq 0$, and for all λ_i and everywhere in $V + S + Z$ the inequality

$$\sum_{i,j=1}^n a_{ij} \lambda_i \lambda_j \geq m \sum_{i=1}^n \lambda_i^2, \quad m = \text{const} > 0.$$

Concerning the boundary of the domain G we shall assume that it has the following property: for every point $P \in S$ there exists a sphere K_P such that $K_P \subset V$ and $\overline{K_P} \cap S = \{P\}$. If the boundary of the domain G is formed by a twice continuously differentiable surface, then this condition is obviously satisfied. By l we denote the direction which forms an acute angle with the inward normal of the sphere K_P at the point P .

Theorem 1. Let $u(X, t)$ be, in the domain V , a solution of the equation

$$L(u) = f(X, t), \tag{1}$$

continuous on $V + S + Z$. Let $(X_1, t_1) \in S$; $u(X, t) \geq u(X_1, t_1)$ on $V + S + Z$. If $f \leq 0$, $u(X_1, t_1) < 0$, and if u is not constant in V_{t_1} , then

$$\lim_{(X,t) \rightarrow (X_1,t_1)} -\frac{u(X,t) - u(X_1,t_1)}{\sqrt{|X - X_1|^2 + (t - t_1)^2}} > 0, \quad (2)$$

where (X, t) varies in the direction l .

Remark 1. From Theorem 1 it follows immediately that a nonconstant (in the domain V_{t_0}) solution u of the equation

$$L(u) = 0 \quad (3)$$

cannot attain either a positive maximum or a negative minimum at the point $(X_0, t_0) \in S$, if $\partial u(X_0, t_0)/\partial l = 0$.

Remark 2. If $c \equiv 0$, then u cannot attain either a maximum or a minimum at the point (X_0, t_0) . This assertion contains a generalization of some theorems proved by Adler and Friedman (4-6).

We shall preface the proof of Theorem 1 with the following lemma.

Lemma. Let K be a sphere of radius R , $K \subset V$. Suppose that at the point (X_1, t_1) , lying on its boundary, u assumes its least negative value. If the point (X_1, t_1) is not a pole* of the sphere K and if

$$u(X, t) > u(X_1, t_1) \quad (4)$$

in \bar{K} for $(X, t) \neq (X_1, t_1)$, then inequality (2) is valid.

Proof. Place the origin of coordinates at the center of the sphere K . Construct a sphere K_1 with center at the point (X_1, t_1) so that the function $g(X) = |X|^2$ attains a positive minimum on the set $\bar{K} \cap \bar{K}_1$. For the function $h = e^{-\alpha(|X|^2 + t^2)} - e^{-\alpha R^2}$ we shall have $L(h) > 0$, if we take α sufficiently large. Therefore the function $u - \varepsilon h$ cannot attain a minimum on $K \cap K_1$; consequently, for sufficiently small ε we shall have $u - \varepsilon h \geq u(X_1, t_1)$ in $\bar{K} \cap \bar{K}_1$, whence the assertion of the lemma is already easily obtained.

Proof of Theorem 1. For every point $(X_2, t_2) \in K_P$ for which $u(X_2, t_2) = u(X_1, t_1)$, it must be, by Nirenberg's theorem (7), that $t_2 < t_1$; thus, one may assume that all points of K_P satisfy inequality (4). To finish the proof, it is now enough to use the preceding lemma.

Theorem 2. Let u be a solution of equation (3), continuous on $V + S + Z$, satisfying the boundary condition

$$l(u) = a \frac{\partial u}{\partial l} + bu = 0 \quad (5)$$

on S and the initial condition

$$u = 0 \tag{6}$$

on Z . Suppose further,** $a^2(P) + b^2(P) \geq k > 0$, $a \geq 0$, $b \leq 0$. Then $u \equiv 0$ in the domain V .

For the proof of Theorem 2 it is enough to use Theorem 1. Let us make one more assumption on the surface S , namely, that it has a continuously varying tangent plane. Then the theorems below can be proved by the same methods as those used by O. A. Oleinik in the work ⁽³⁾.

Theorem 3. Let $u(P)$ be a solution of equation (1), continuous on $V + S + Z$, satisfying conditions (5) and (6). Then, if we denote

$$M = \sup_{V+S+Z} |f|,$$

then

$$|u(P)| \leq \frac{M}{\inf |c|}.$$

Theorem 4. Suppose that for any functions $\psi_1(P), \psi_2(P)$ of class C_k ($k \geq 0$) there exists a solution of the equation $L(u) = 0$, satisfying the conditions $u = \psi_1$ on Z , $l(u) = \psi_2$ on S . If the functions a, b belong to the class C_k , then the solution of equation (1), satisfying the conditions $u = \varphi_1$ on Z and $l(u) = \varphi_2$ on S , satisfies the inequality

$$|u(P)| \leq \frac{M}{\inf |c|} + m_1 + \frac{m_2}{\inf(|a| + |b|)} \left[K_1 + \frac{K_2}{\inf |c|} \sup R \right],$$

where

$$R = \sum_{i,k=1}^n |a_{ik}| + \sum_{i=1}^n |b_i| + |c| + 1, \quad M = \sup |f|, \quad m_1 = \sup |\varphi_1|, \quad m_2 = \sup |\varphi_2|.$$

* We call a point of the sphere K a pole if its ordinate t is the smallest or the largest among the ordinates t of all points of the sphere K .

** Everywhere below these assumptions remain in force.

Theorem 5. Let us leave unchanged the notation and assumptions of Theorem 4. Further, let $\sup c < 0$. Let \bar{u} be a solution of the equation

$$L_1(\bar{u}) = \sum_{i,j=1}^n \bar{a}_{ij} \bar{u}_{x_i x_j} + \sum_{i=1}^n \bar{b}_i \bar{u}_{x_i} - \bar{u}_t + \bar{c} \bar{u},$$

satisfying the conditions $\bar{u} = \varphi_1$ on Z and $\bar{a} \frac{\partial \bar{u}}{\partial l} + \bar{b} \bar{u} = \varphi_2$ on S , and having two continuous and bounded derivatives on $V + S + Z$. Then the difference $|u - \bar{u}|$ will be arbitrarily small provided that the differences of the coefficients of the equations, the differences of the initial and boundary conditions, and the differences of the right-hand sides of the equations are sufficiently small in absolute value.

Theorem 6. Let, for a function W of class C_2 on \bar{V} , for which the inequality $\min_{\bar{V}} L(W) > 0$ holds, there exist a solution u_W of the equation $L(u_W) = L(W)$, satisfying homogeneous initial and boundary conditions, and let, further, there exist a solution u_2 of equation (3) such that $u_2 = \varphi_1$ on Z and $l(u_2) = \varphi_2$ on S . Then for the solution u of equation (1), for which the same initial and boundary conditions are fulfilled as for the function u_2 , we have the estimate

$$|u(P)| \leq m_1 + \frac{m_2}{\inf |b|} + CM,$$

where C is a constant depending on the coefficient of the equation and on the functions a, b , but depending neither on φ_i nor on f .

Theorems analogous to Theorems 1-6 can also be proved in the case of a smooth moving boundary S , whose tangent plane is at no point perpendicular to the t -axis.

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