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Abstract

Full Text

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Second-Order Invariants of a Quaternary Differential Quadratic Form

(Presented by Academician I. M. Vinogradov on 19 XI 1956)

MATHEMATICS

1°. The investigations of É. Cartan ([1]) and N. A. Rosenson ([2]) do not exhaust the problem of finding a basis for the complete system of scalar invariants of second order of a differential quadratic form in 4 variables: in ([1]) the case of a quaternary differential form of signature $s = -2$ is considered, and in this case the problem mentioned is reduced to one still unsolved question in the theory of algebraic invariants (([1]), p. 31); article ([2]) is unsatisfactory—for example, the proof of the invariance of the functions M_s on p. 76 is erroneous.

2°. On the basis of the reduction theorem, every scalar differential invariant of second order of a manifold V_n is a simultaneous algebraic invariant of the fundamental tensor and the second affine extension of the latter. Thomas, basing himself on group theory, proved ([3]) that invariants I of this kind satisfy a complete system of n^2 $(n > 2)$ independent homogeneous first-order differential equations

$$\left[\begin{array}{c} \mu \\ \alpha\beta\nu \end{array} \right] \frac{\partial I}{\partial g_{\alpha\beta}} + \left[\begin{array}{c} \mu \\ \alpha\beta\gamma\delta\nu \end{array} \right] \frac{\partial I}{\partial g_{\alpha\beta,\gamma\delta}} = 0 \qquad (\mu, \nu = 1, 2, \ldots, n).$$

From this general proposition follows the existence of 14 absolute scalar differential invariants of second order of four-dimensional Riemannian manifolds. In what follows we shall regard I_i as algebraic invariants of the fundamental tensor $g_{\alpha\beta}$ and of its curvature tensor $B_{\alpha\beta\gamma\delta}$.

Further, if, in accordance with Cartan' s geometric conception, one imagines at each point of V_4 a tangent Euclidean or pseudo-Euclidean space of 4 dimensions, referred to rectangular Cartesian coordinates, then the invariants sought will prove to be orthogonal invariants of the curvature tensor. Passing to the search for them precisely in this last formulation of the problem, we must distinguish, depending on the signature s of the fundamental form, three essentially different cases: $s = 4, 0, -2$.

Let us take the group $O^+(4)$, which leaves invariant the sum of squares of 4 variables, and see what transformations are induced on the coefficients of the homogeneous polynomial

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$$B_{\alpha\beta\gamma\delta}u^\alpha\tilde{u}^\beta u^\gamma\tilde{u}^\delta \qquad (\alpha,\beta,\gamma,\delta=1,\dots,4),$$

when the variables u^i are subjected to substitutions of this group. It is not difficult to find the infinitesimal operators of these induced transformations. To write them explicitly, let us agree on the notation for the essential components of the tensor $B_{\alpha\beta\gamma\delta}$:

$$B_{1212}=x_1,\quad B_{1313}=x_2,\quad B_{1414}=x_3,\quad B_{2323}=x_4,\quad B_{2424}=x_5,\quad B_{3434}=x_6,$$
$$B_{1213}=y_1,\quad B_{1214}=y_2,\quad B_{1223}=y_3,\quad B_{1224}=y_4,\quad B_{1314}=y_5,\quad B_{1323}=y_6,$$
$$B_{1334}=y_7,\quad B_{1424}=y_8,\quad B_{1434}=y_9,\quad B_{2324}=y_{10},\quad A_{2334}=y_{11},\quad B_{2334}=y_{12},$$
$$B_{1234}=z_1,\qquad B_{1324}=z_2,\qquad B_{1423}=z_3,\quad z_1-z_2+z_3=0.$$

The invariants I_i are the roots of the linear homogeneous differential operators given in Table 1.

In an analogous manner, the corresponding systems of operators are composed in the two other cases.

3°. In all three cases the finding of invariants reduces to the integration of the following complete systems of linear differential equations, in which the variables a_i, b_i, c_i, depending on the value of the signature of the principal form, are specifically expressed in terms of x_i, y_k, z_j:

I.

$$A_1f\equiv 2a_4p_1-2a_4p_2+(a_2-a_1)p_4+a_6p_5-a_5p_6=0;$$

$$A_2f\equiv -2a_5p_1+2a_5p_3-a_6p_4+(a_1-a_3)p_5+a_4p_6=0;$$

$$A_3f\equiv 2a_6p_2-2a_6p_3+a_5p_4-a_4p_5+(a_3-a_2)p_6=0$$

$$\left(p_i=\frac{\partial f}{\partial a_i}\right).$$

The variables a_i are connected by one relation.

II. $B_if=0$ $(i=1,2,3)$. The operators B_if are obtained from the operators A_if by the simple replacement of a_i by b_i.

III.

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$$C_1 f \equiv (c_2+c_4)q_1+(c_5-c_1)q_2+c_6q_3+(c_5-c_1)q_4-(c_2+c_4)q_5-c_3q_6+c_8q_7-c_7q_8 = 0;$$

$$C_2 f \equiv -(c_3+c_7)q_1-c_8q_2+(c_1-c_9)q_3-c_6q_4+c_4q_6+(c_1-c_9)q_7+c_2q_8+(c_3+c_7)q_9 = 0;$$

$$C_3 f \equiv -c_3q_2+c_2q_3+c_7q_4+(c_8-c_6)q_5+(c_5+c_9)q_6-c_4q_7-(c_5+c_9)q_8+(c_8-c_6)q_9 = 0;$$

$$C_4 f \equiv c_3q_2-c_2q_3+c_7q_4+(c_8+c_6)q_5+(c_9-c_5)q_6-c_4q_7+(c_9-c_5)q_8-(c_8+c_6)q_9 = 0;$$

$$C_5 f \equiv (c_7-c_3)q_1+c_8q_2+(c_1+c_9)q_3-c_6q_4+c_4q_6-(c_1+c_9)q_7-c_2q_8+(c_7-c_3)q_9 = 0;$$

$$C_6 f \equiv (c_4-c_2)q_1+(c_1+c_5)q_2+c_6q_3-(c_1+c_5)q_4+(c_4-c_2)q_5-c_3q_6-c_8q_7+c_7q_8 = 0$$

$$\left(q_j = \frac{\partial f}{\partial c_j}\right).$$

IV.

$$(A_1 + C_1)f = 0, \qquad (A_2 + C_2)f = 0, \qquad (A_3 + C_3)f = 0,$$

$$(A_3 + C_4)f = 0, \qquad (A_2 - C_5)f = 0, \qquad (A_1 + C_6)f = 0.$$

V.

$$(B_1 + C_1)f = 0, \qquad (B_2 + C_2)f = 0, \qquad (-B_3 + C_3)f = 0,$$

$$(B_3 + C_4)f = 0, \qquad (B_2 + C_5)f = 0, \qquad (-B_1 + C_6)f = 0.$$

The integration of system I presents no difficulty. As a complete system of 3 equations in 5 independent variables, it has 2 integrals:

$$\sigma_1 \equiv a_1a_2 + a_1a_3 + a_2a_3 - a_4^2 - a_5^2 - a_6^2;$$

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$$\sigma_2 \equiv a_1a_2a_3 + 2a_4a_5a_6 - a_2a_5^2 - a_1a_6^2 - a_3a_4^2.$$

Table 1

Operator	$\frac{\partial f}{\partial x_1}$	$\frac{\partial f}{\partial x_2}$	$\frac{\partial f}{\partial x_3}$	$\frac{\partial f}{\partial x_4}$	$\frac{\partial f}{\partial x_5}$	$\frac{\partial f}{\partial x_6}$	$\frac{\partial f}{\partial y_1}$	$\frac{\partial f}{\partial y_2}$	$\frac{\partial f}{\partial y_3}$	$\frac{\partial f}{\partial y_4}$	$\frac{\partial f}{\partial y_5}$	$\frac{\partial f}{\partial y_6}$	$\frac{\partial f}{\partial y_7}$	$\frac{\partial f}{\partial y_8}$	$\frac{\partial f}{\partial y_9}$	$\frac{\partial f}{\partial y_{10}}$	$\frac{\partial f}{\partial y_{11}}$	$\frac{\partial f}{\partial y_{12}}$	$\frac{\partial f}{\partial z_1}$	$\frac{\partial f}{\partial z_2}$	$\frac{\partial f}{\partial z_3}$
Z_1	0	$-2y_6$	$2y_5$	$2y_6$	$2y_8$	0	$-y_3$	$-y_4$	y_1	$+y_2$	$-z_2$	y_2	$-y_1$	x_3	$-y_{12}$	z_3	$+z_1$	y_9	0	y_5	$-y_5-y_{10}$
Z_2	$2y_3$	0	$-2y_4$	$-2y_3$	$2y_9$	y_6	z_1	$+y_5$	$-y_6$	y_1	$-x_1$	$+y_5$	$-y_{12}$	x_3	$-y_4$	z_1	$+y_{12}$	y_2	$+0$	$-y_2+y_{11}$	
Z_3	$2y_4$	$2y_7$	0	0	$-2y_{12}$	$-2y_8$	z_3	$+y_8$	y_7	$-y_1$	y_3	y_1	$-x_1$	x_6	z_1	z_3	$-y_4$	x_6	$-y_4$	y_{11}	y_2+y_{12}	$-y_4+y_5+y_{12}$
Z_4	$-2y_2$	$2y_1$	0	0	$-2y_{11}$	$2y_5$	$-y_8$	y_7	y_9	$-y_2$	y_3	$-x_5$	$+y_9$	y_5	$-y_{11}$	x_6	$+y_4$	$-y_{10}$	$-y_1$	y_{12}	y_4+y_{12}	$-y_8+y_{11}$
Z_5	$-2y_1$	$2y_2$	$2y_{12}$	$2y_{11}$	$-2y_{10}$	$-2y_{12}$	x_2	$-y_5$	y_3	$-y_8$	$-y_9$	y_1	$-y_6$	y_9	$-y_5$	$-y_{11}$	y_{12}	$-y_{10}$	$-y_3$	y_4	$y_{12}+y_4$	$+y_4+y_{11}$
Z_6	0	$-2y_5$	$-2y_4$	$-2y_{10}$	0	$-y_2$	y_1	x_1	$-y_8$	$-y_3$	$-z_2$	y_9	z_2	$+y_7$	x_4	$-y_{12}$	y_{11}	0	y_4	$-y_4-y_8$	y_8	

Two functionally independent solutions χ_1, χ_2 of system II are obtained from σ_1, σ_2 by replacing a_i in them by b_i.

Solving system III is technically much more difficult: here the question is that of integrating a closed system of 6 equations with 9 independent variables. Its solutions are nevertheless obtained by successive application of the known methods of integrating linear systems. We find:

$$\theta_1 \equiv \sum_{i=1}^{9} c_i^2;$$

$$\theta_2 \equiv c_1c_5c_9 + c_3c_4c_8 + c_2c_6c_7 - c_1c_6c_8 - c_2c_4c_9 - c_3c_5c_3;$$

$$\begin{aligned}\theta_3 \equiv {} & (c_1^2 + c_2^2 + c_3^2)(c_4^2 + c_5^2 + c_6^2) + (c_1^2 + c_2^2 + c_3^2)(c_7^2 + c_8^2 + c_9^2) + \\ & + (c_4^2 + c_5^2 + c_6^2)(c_7^2 + c_8^2 + c_9^2) - (c_1c_4 + c_2c_5 + c_3c_6)^2 - \\ & - (c_1c_7 + c_2c_8 + c_3c_9)^2 - (c_4c_7 + c_5c_8 + c_6c_9)^2.\end{aligned}$$

Turning to system IV, we note that 5 of its 8 integrals are already known to us. The 3 unknown integrals of this system must be functions of both sets of variables. They can be sought by the usual methods. We obtain:

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$$\omega_1 = a_1(c_4^2+c_5^2+c_6^2+c_7^2+c_8^2+c_9^2)+a_2(c_1^2+c_2^2+c_3^2+c_7^2+c_8^2+c_9^2)+ \\ +a_3(c_1^2+c_2^2+c_3^2+c_4^2+c_5^2+c_6^2)-2a_4(c_1c_4+c_2c_5+c_3c_6)- \\ -2a_5(c_1c_7+c_2c_8+c_3c_9)-2a_6(c_4c_7+c_5c_8+c_6c_9);$$

$$\omega_2 = (c_1^2+c_2^2+c_3^2)(a_2a_3-a_6^2)+(c_4^2+c_5^2+c_6^2)(a_1a_3-a_5^2)+ \\ +(c_7^2+c_8^2+c_9^2)(a_1a_2-a_4^2)+2(c_1c_4+c_2c_5+c_3c_6)(a_5a_6-a_3a_4)+ \\ +2(c_1c_7+c_2c_8+c_3c_9)(a_4a_6-a_2a_5)+2(c_4c_7+c_5c_8+c_6c_9)(a_4a_5-a_1a_6);$$

$$\omega_3 = (a_1c_1+a_4c_4+a_5c_7)^2+(a_1c_2+a_4c_5+a_5c_8)^2+(a_1c_3+a_4c_6+a_5c_9)^2+ \\ +(a_4c_1+a_2c_4+a_6c_7)^2+(a_4c_2+a_2c_5+a_6c_8)^2+(a_4c_3+a_2c_6+a_6c_9)^2+ \\ +(a_5c_1+a_6c_4+a_3c_7)^2+(a_5c_2+a_6c_5+a_3c_8)^2+(a_5c_3+a_6c_6+a_3c_9)^2.$$

Proceeding in the same way as above, we find three mixed integrals of the system $\bar{y}$:

$$\bar{\omega}_1 \equiv b_1(c_2^2+c_3^2+c_5^2+c_6^2+c_8^2+c_9^2)+b_2(c_1^2+c_3^2+c_4^2+c_6^2+c_7^2+c_9^2)+ \\ +b_3(c_1^2+c_2^2+c_4^2+c_5^2+c_7^2+c_8^2)-2b_4(c_1c_2+c_4c_5+c_7c_8)- \\ -2b_5(c_1c_3+c_4c_6+c_7c_9)-2b_6(c_2c_3+c_5c_6+c_8c_9);$$

$$\bar{\omega}_2 \equiv (b_2b_3-b_6^2)(c_1^2+c_4^2+c_7^2)+(b_1b_3-b_5^2)(c_2^2+c_5^2+c_8^2)+ \\ +(b_1b_2-b_4^2)(c_3^2+c_6^2+c_9^2)+2(b_5b_6-b_3b_4)(c_1c_2+c_4c_5+c_7c_8)+ \\ +2(b_4b_6-b_2b_5)(c_1c_3+c_4c_6+c_7c_9)+2(b_4b_5-b_1b_6)(c_2c_3+c_5c_6+c_8c_9);$$

$$\bar{\omega}_3 \equiv (c_1b_1+c_2b_4+c_3b_5)^2+(c_4b_4+c_5b_2+c_6b_6)^2+(c_7b_5+c_8b_6+c_9b_3)^2+ \\ +(c_1b_4+c_2b_2+c_3b_6)^2+(c_1b_5+c_2b_6+c_3b_3)^2+(c_4b_1+c_5b_4+c_6b_5)^2+ \\ +(c_4b_5+c_5b_6+c_6b_3)^2+(c_7b_1+c_8b_4+c_9b_5)^2+(c_7b_4+c_8b_2+c_9b_6)^2.$$

Investigation of the rank of the Jacobian matrix of all the invariants constructed above shows their functional independence.

Thus, the following theorem holds.

Theorem 1. The set of 14 functions: B (the total curvature), $\sigma_i, x_i, \theta_k, \omega_k, \overline{\omega}_k$ $(i=1,2;\ k=1,2,3)$ forms the simplest basis of the complete system of scalar differential invariants of the second order of a quaternary differential quadratic form (of the manifold V_4).

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We formulate here, in particular, two special cases of Theorem 1, which, in the author' s opinion, have independent significance for the theory of differential invariants of generalized spaces.

Theorem 2. The scalars $B, \theta_1, \theta_2, \theta_3$ give a basis of the complete system of scalar differential invariants of the second order of conformal-Riemannian spaces of 4 dimensions.

Theorem 3. The simplest system of basic differential invariants of the second order of a 4-dimensional Einstein space consists of the invariants $\sigma_1, \sigma_2, x_1, x_2$.

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Note: Figure translations are in progress. See original paper for figures.

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