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Abstract

Full Text

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ON CONDITIONS DETERMINING OBJECTS OF AFFINE CONNECTION OF A RIEMANNIAN SPACE

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As is known, a given affine connection can be regarded as associated with a Riemannian or pseudo-Riemannian metric if and only if there exists, in this connection, a parallel field of the metric tensor g_{ij}. The question of the existence of such a field reduces to the study of the integrability conditions of the equations

$$Dg_{ij} = 0 \qquad (i, j = 1, \dots, n), \tag{1}$$

where Dg_{ij} is the absolute differential of the metric tensor.

Below these conditions are given for the case $n = 3$; the smoothness class of the manifold X_3 and of the given affine connection is assumed sufficiently high.

Let $R_{ij}, \nabla_k R_{ij}, \nabla^2_{kl} R_{ij}$ $(i, j, k, l = 1, 2, 3)$ be the matrices formed from the components of the curvature tensor and its covariant derivatives of the first and second orders (for example, the matrix R_{ij} is composed of the components $R^l_{ij,k}$ of the curvature tensor, where k, l are the numbers of the matrix element; i, j are the numbers of the matrix).

If the given connection is Riemannian, then in the basic manifold X_3 there exists a closed set without interior points, decomposing X_3 into open connected sets, in each of which one of the following conditions is satisfied:

1) $R_{ij}(x) = 0$ $(i, j = 1, 2, 3)$ at each point x.

2) Among the matrices R_{ij} there exists one $A(x)$, different from zero at each point; all the remaining matrices among $R_{ij}, \nabla_k R_{ij}$ $(i, j, k = 1, 2, 3)$ differ from it only by scalar multipliers.

3) Among the matrices R_{ij} there exists one $A(x)$, different from zero at each point; the other two matrices among R_{ij} differ from it only by scalar multipliers, but among the matrices $\nabla_k R_{ij}$ there exists a matrix $B(x)$, linearly independent of $A(x)$. All matrices among $\nabla_k R_{ij}, \Delta^2_{kl} R_{ij}$ are linear combinations of the matrices $A(x), B(x)$, and $C(x) = A(x)B(x) - B(x)A(x)$.

4) Among the matrices R_{ij} one can choose two linearly independent matrices $A(x), B(x)$; all the matrices $R_{ij}, \nabla_k R_{ij}$ are linear combinations of the matrices A, B, and $C = AB - BA$.

Finally, at each point of X_3 the following condition is satisfied: a) the matrix $A(x)$ mentioned in condition 2) is similar to a skew-symmetric one; the matrices $A(x)$ and $B(x)$ mentioned in conditions 3), 4) can be simultaneously reduced to skew-symmetric form.

Theorem 1. *If in some domain $V \subset X_3$ one of the conditions 1), 2), 3), 4) is satisfied together with condition a), then there exists a field of a positive-definite metric tensor g_{ij} satisfying equations (1) in this domain.*

We note that in the case of a pseudo-Riemannian metric condition a) must be modified accordingly.

The proof of this theorem is based on the following assertion concerning the holonomy groups of composite manifolds with a linear connection. Let $L_k(x)$ be the minimal Lie algebra containing the matrices formed from the components of the curvature tensor and its covariant derivatives up to order k, inclusive, at the point x. Let $p(x)$ be the smallest number such that $L_p(x) = L_{p+1}(x)$.

Theorem 2. *There exists a set M, containing no interior points, which decomposes the base manifold of a space with a linear connection into domains such that, for each of them, the Lie algebra determining the identity component of the holonomy group of this domain coincides with $L_p(x)$ at an arbitrary point x.*

This theorem is a very simple refinement of a known result of Nijenhuis (1); for its proof it is enough to observe that the algebra $L_p(x)$ is constant (for the given connection) in some domain if $p(x)$ and the dimension of $L_p(x)$ are constant in it.

Finally, we note that the last theorem makes it possible to establish the integrability condition for equations (1) also in the general case of arbitrary n, but the algebraic form of these conditions is very complicated.

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Note: Figure translations are in progress. See original paper for figures.