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Abstract

Full Text

MATHEMATICS

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SINGULAR INTEGRALS IN THE SPACES L_p

(Presented by Academician V. I. Smirnov, 20 V 1957)

In the present note we use the notation and terminology of the papers ^(1,2).

1. **Theorem 1.** *If the symbol $\Phi(x, \vartheta)$ of the singular operator*

$$Au = a(x)u(x) + \int_{E_m} \frac{f(x, \theta)}{r^m} u(y) dy \quad (1)$$

and the derivatives of this symbol of the form

$$\frac{\partial \Phi}{\partial \vartheta_1}, \quad \frac{\partial^2 \Phi}{\partial \vartheta_1 \partial \vartheta_2}, \quad \dots, \quad \frac{\partial^{m-2} \Phi}{\partial \vartheta_1 \dots \partial \vartheta_{m-2}} \quad (2)$$

are continuous for fixed x and bounded independently of x , while the derivative

$$\frac{\partial^{m-1} \Phi}{\partial \vartheta_1 \dots \partial \vartheta_{m-2} \partial \vartheta_{m-1}} \quad (3)$$

exists as a generalized derivative and satisfies the inequality

$$\int_0^\pi \dots \int_0^\pi \int_0^{2\pi} \left| \frac{\partial^{m-1} \Phi}{\partial \vartheta_1 \dots \partial \vartheta_{m-2} \partial \vartheta_{m-1}} \right|^{p'} d\vartheta_1 \dots d\vartheta_{m-2} d\vartheta_{m-1} \leq C = \text{const}, \quad 1 < p' < \infty, \quad (4)$$

then the operator A is bounded in $L_p(E_m)$, $\frac{1}{p} + \frac{1}{p'} = 1$.

The operator A , which for the time being we shall consider on the set of infinitely differentiable finite functions dense in $L_p(E_m)$, can be represented in the form (1)

$$Au = \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} \Phi(x, \vartheta) d\mathcal{E}_1(\vartheta_1) \dots d\mathcal{E}_{m-2}(\vartheta_{m-2}) d\mathcal{E}_{m-1}(\vartheta_{m-1}) u. \quad (5)$$

By integration by parts we bring formula (5) to the form

$$Au = \sum_{k=0}^m \pm \int \cdots \int \frac{\partial^k \Phi}{\partial \vartheta_{i_1} \cdots \partial \vartheta_{i_k}} \mathcal{E}_{i_1}(\vartheta_1) \cdots \mathcal{E}_{i_s}(\vartheta_s) u d\vartheta_{j_1} \cdots d\vartheta_{j_s}. \quad (6)$$

Under the integral signs in (6) there occur only derivatives of the form (2) or (3), and the derivative (3) occurs only under an integral sign of multiplicity $m - 1$. Applying Hölder's inequality, we see that Theorem 1 will be proved if it is established that the operators $\mathcal{E}_j(\vartheta_j)$ are uniformly bounded in $L_p(E_m)$.

It is not difficult to verify that $\mathcal{E}_j(\vartheta_j^{(0)})$ is a singular operator whose symbol is equal to one for $0 \leq \vartheta_j \leq \vartheta_j^{(0)}$ and to zero for $\vartheta_j > \vartheta_j^{(0)}$. Denote by $\Phi_j(x)$ the function that is constant on rays issuing from the initial-

of the coordinates, and on the unit sphere coincides with the just-mentioned symbol. Then [2]

$$\mathcal{E}_j(\vartheta_j^{(0)})u = \frac{1}{(2\pi)^{m/2}} \int_{E_m} e^{i(x,y)} \Phi_j(y) \tilde{u}(y) dy, \quad (7)$$

where \tilde{u} denotes the Fourier transform of the function u . To simplify the notation, we shall carry out the subsequent computations for the case $m = 2$.

Expand $u(x)$ in a Fourier series in the square $-l \leq x_1, x_2 \leq l$, where l is an arbitrary number:

$$u(x) = \sum_{\nu_1, \nu_2 = -\infty}^{+\infty} \frac{1}{4l^2} \int_{-l}^l \int_{-l}^l u(t) \exp \left\{ \frac{i\pi}{l} [\nu_1(x_1 - t_1) + \nu_2(x_2 - t_2)] \right\} dt. \quad (8)$$

Put also

$$v_1(x) = \sum_{\nu_1, \nu_2 = -\infty}^{+\infty} \frac{\lambda(\nu_1, \nu_2)}{4l^2} \int_{-l}^l \int_{-l}^l u(t) \exp \left\{ \frac{i\pi}{l} [\nu_1(x_1 - t_1) + \nu_2(x_2 - t_2)] \right\} dt,$$

$$\lambda(\nu_1, \nu_2) = \Phi \left(\frac{\pi\nu_1}{l}, \frac{\pi\nu_2}{l} \right). \quad (9)$$

The transformation of $u(x)$ into $v_1(x)$ consists in the fact that the term of the series (8) with indices ν_1, ν_2 is left unchanged if the point (ν_1, ν_2) lies inside or on the sides of the angle $0 \leq \vartheta_1 \leq \vartheta_1^{(0)}$, and is deleted otherwise.

Introduce the notation $\exp\left(\frac{i\pi x_2}{l}\right) = z$, $u(x) = u(x_1, x_2) = f(z)$; when x_2 runs over the segment $[-l, l]$, the point z runs over the circle $|z| = 1$. Consider the operator

$$Qf = \frac{1}{2}f(z) + \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} d\zeta. \quad (10)$$

By the known theorem of Riesz, this operator is bounded in L_p , $1 < p < \infty$, so that

$$\int_{|z|=1} |Qf|^p |dz| \leq A_p^p \int_{|z|=1} |f|^p |dz|, \quad (11)$$

where A_p is a constant depending only on p .

Return to the variable x_2 . The action of the operator Q consists in deleting from the series (8) the terms with $\nu_2 < 0$. Keeping the notation Q for this operator and returning to the variable x_2 , we have, by virtue of inequality (11),

$$\int_{-l}^l |Qu(x_1, x_2)|^p dx_2 \leq A_p^p \int_{-l}^l |u(x_1, x_2)|^p dx_2. \quad (12)$$

Let us turn to the transformation (9). If $\vartheta_1^{(0)} = 0$ or $\vartheta_1^{(0)} = 2\pi$, then the boundedness of this transformation is obvious; if $\vartheta_1^{(0)} = \pi/2, \pi, 3\pi/2$, then this boundedness follows easily from (12). This circumstance permits us to restrict ourselves to the case $0 < \vartheta_1^{(0)} < \pi/2$; at the same time one may assume that the Fourier coefficients of the function $u(x_1, x_2)$ are nonzero only for $\nu_1 \geq 0, \nu_2 \geq 0$. It is then not difficult to see that

$$v_1(x_1, x_2) = u(x_1, x_2) - \exp\{i[x_1 \tan \vartheta_1^{(0)}]x_2\} Q\{\exp\{-i[x_1 \tan \vartheta_1^{(0)}]x_2\} u(x_1, x_2)\}. \quad (13)$$

From formula (13) there follows the boundedness, uniform with respect to $\vartheta_1^{(0)}$ and l , of the transformation (9) in the space $L_p(-l, l)$; there exists a constant B_p , depending only on p , such that the inequality

$$\int_{-l}^l \int_{-l}^l |v_l(x)|^p dx \leq B_p^p \int_{-l}^l \int_{-l}^l |u(x)|^p dx$$

holds.

We shall strengthen this inequality by integrating on the left over the square $-N \leq x_1, x_2 \leq N$, $N \leq l$, and on the right over the whole plane:

$$\int_{-N}^N \int_{-N}^N |v_l(x)|^p dx \leq B_p^p \iint_{E_2} |u(x)|^p dx. \quad (14)$$

Repeating, with slight changes, the arguments of the paper ⁽³⁾, we find that in the square $-N \leq x_1, x_2 \leq N$ the function $v_l(x)$ tends uniformly to $\mathcal{E}_1(\vartheta_1^{(0)})u$. Now putting in (14) first $l \rightarrow \infty$, and then $N \rightarrow \infty$, we find that the operator $\mathcal{E}_1(\vartheta_1^{(0)})$ is uniformly bounded in $L_p(E_2)$; as was already said, theorem 1 follows from this.

Theorem 1 remains valid for any singular operator of the form (5), even if it is not representable in the form (1). This may occur, for example, in the case when the symbol is a sufficiently many times continuously differentiable function of the angular coordinates $\vartheta_1, \vartheta_2, \dots, \vartheta_{m-1}$, but is discontinuous as a function of a point of the unit sphere.

For $m = 2$ theorem 1 coincides with theorem 2 of Calderon and Zygmund ⁽⁴⁾; for $m > 2$ there are cases which are covered by the theorems of the paper ⁽⁴⁾, but are not covered by our theorem 1, and conversely.

Theorem 1 is extended in an obvious way to the case where the integration in (1) is carried out not over the Euclidean space E_m , but over an arbitrary closed Lyapunov manifold Γ of m dimensions.

2. Consider the singular integral equation

$$A_1 u \equiv a(x)u(x) + \int_{E_m} \frac{f(x, \theta)}{r^m} u(y) dy + Tu = g(x); \quad g(x) \in L_p(E_m). \quad (15)$$

We make the following assumptions: 1) the operator T is completely continuous in $L_p(E_m)$; 2) the symbol of the operator A_1 satisfies the conditions of theorem 1 and nowhere vanishes; 3) the coefficient $a(x)$ and the characteristic $f(x, \theta)$ satisfy the inequalities

$$|a(x) - a(y)| \leq Mr^\gamma (1 + |x|^2)^{m(\frac{1}{r} - \frac{1}{p}) - \frac{\gamma}{2}} (1 + |y|^2)^{m(\frac{1}{2} - \frac{1}{p'}) - \frac{\gamma}{2}}, \quad (16)$$

$$|f(x, \theta) - f(y, \theta)| \leq Mr^\gamma (1 + |x|^2)^{m(\frac{1}{2} - \frac{1}{p}) - \frac{\gamma}{2}} (1 + |y|^2)^{m(\frac{1}{2} - \frac{1}{p'}) - \frac{\gamma}{2}},$$

where M and γ are constants. $M > 0$, $0 < \gamma \leq 1$.

The index* of the operator with symbol $e^{2i\vartheta_j}$ ($j \leq m - 2$) or $e^{i\vartheta_{m-1}}$ in the space $L_p(E_m)$ is equal to zero, as is easily seen at least from the following

considerations. In the given case the adjoint operator has as symbol the function $e^{-2i\vartheta_j}$ or $e^{-i\vartheta_{m-1}}$, but in the same way one can make the symbol of the given—

* The index of an operator A is called the difference between the numbers of linearly independent solutions of

operator under a suitable change of coordinates*. Further, the theorem on the normal solvability of an operator admitting regularization, which was proved by the author ⁵ for the case of Hilbert space, is easily carried over to the case of an arbitrary Banach space. The facts just noted make it possible to apply to equation (15), regarded as an equation in the space $L_p(E_m)$, the arguments of the paper ². This leads to the following theorems.

Theorem 2. *The operator A_1 is normally solvable in $L_p(E_m)$.*

Theorem 3. *The equation $A_1 u = 0$ has in $L_p(E_m)$ only a finite number of linearly independent solutions. The index of the operator A_1 is equal to zero in $L_p(E_m)$.*

It follows from Theorems 2 and 3, as usual, that for equation (15) in $L_p(E_m)$ the Fredholm alternative holds.

Theorems 2 and 3 can also be extended to the case where integration in (15) is performed over a closed differentiable manifold; conditions (16) in this case are replaced by the requirement that a and f satisfy the Hölder condition.

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- ⁴ A. P. Calderon, A. Zygmund, *Am. J. Math.*, 78, No. 2 (1956).
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* Using this consideration, one can simplify the proof of Theorem 1.7, which is fundamental for the paper ².

Note: Figure translations are in progress. See original paper for figures.

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