MATHEMATICS

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Abstract

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ON THE CAUCHY PROBLEM FOR LINEAR ELLIPTIC EQUATIONS OF SECOND ORDER

(Presented by Academician S. L. Sobolev on 21 VIII 1956)

As is known, the Cauchy problem for elliptic equations is ill-posed (Hadamard' s example). However, as Carleman first showed, the Cauchy problem for a second-order elliptic equation in the plane is stable in the class of bounded solutions. Carleman obtained estimates characterizing this stability. Estimates characterizing the stability of the Cauchy problem in space were obtained by the author for the Laplace equation (^{1}), and by E. M. Landis for an arbitrary linear elliptic equation with sufficiently smooth coefficients (^{2}).

We state a theorem generalizing the author' s results (^{1}) to the case of a linear elliptic equation of a somewhat different form than that considered by E. M. Landis.

Theorem 1. Let $u(t,x)$ (x here is an n-dimensional vector) be a function satisfying a linear elliptic equation with variable coefficients

$$\frac{\partial^2 u}{\partial t^2}+\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i\frac{\partial u}{\partial x_i}+cu=0 \tag{1}$$

in the straight cylinder $x\in\Omega$, $0\le t\le l$, vanishing for $t=0$ and on the boundary of Ω for every t (Ω is a simply connected bounded domain with smooth boundary).

Concerning the coefficients of the equation it is assumed that:

a) The functions a_{ij} possess derivatives $\partial a_{ij}/\partial x_i$, $\partial^2 a_{ij}/\partial x_i\partial x_j$, $\partial^2 a_{ij}/\partial t^2$ that are square summable over the domain Ω for every t, so that

$$\int_\Omega \left(\frac{\partial a_{ij}}{\partial x_i}\right)^2 dx\le\lambda_1,\qquad \int_\Omega \left(\frac{\partial^2 a_{ij}}{\partial x_i\partial x_j}\right)^2 dx\le\lambda_2,$$

$$\int_\Omega \left(\frac{\partial^2 a_{ij}}{\partial t^2}\right)^2 dx\le\lambda_3.$$

b) The functions b_i and the derivatives $\partial b_i/\partial x_i$ are square summable over the domain Ω, so that

$$\int_\Omega b_i^2\,dx \le \lambda_4, \qquad \int_\Omega \left(\frac{\partial b_i}{\partial x_i}\right)^2 dx \le \lambda_5.$$

c) The function c is square summable, so that

$$\int_\Omega c^2\,dx \le \lambda_6,$$

d)

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geqslant \alpha > 0 \quad \text{for} \quad \sum_{i=1}^{n} \xi_i^2 = 1,$$

where λ_k $(k = 1, \dots, 6)$ and α are certain constants.

Then, if the inequalities

$$\int_\Omega u_y^2(0,x)\,dx \leqslant \varepsilon, \qquad \int_\Omega u^2(l,x)\,dx \leqslant M,$$

hold, then the inequality

$$\int_\Omega u^2(t,x)\,dx \leqslant C_1 M^{C_2\frac{t}{l}} \varepsilon^{C_2\frac{l-t}{l}},$$

also holds, where the constants C_1, C_2 depend on the numbers λ_k and α*.

From Theorem 1, as a consequence, one can obtain the following theorems, refining the results of E. M. Landis.

Theorem 2. Let $u(x)$ be a function satisfying the linear elliptic equation

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = 0 \tag{2}$$

in the unit ball G. Concerning the coefficients of the equation it is assumed that a_{ij} possess continuous derivatives $\partial a_{ij}/\partial x_k$ and $\partial^2 a_{ij}/\partial x_m \partial x_k$, b_i possess continuous derivatives $\partial b_i/\partial x_k$, and that all coefficients and the indicated derivatives are bounded in G by unity. Further, it is assumed that in the domain G

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geqslant \alpha > 0 \quad \text{for} \quad \sum_{i=1}^{n} \xi_i^2 = 1.$$

Let S be the boundary of G and γ_l the part of S determined by the relations

$$\sum_{i=1}^{n} x_i^2 = 1; \qquad \sum_{i=2}^{n} x_i^2 \leqslant l^2; \qquad x_1 > 0.$$

Then there exist constants C_1, C_2, depending only on the dimension n of the space and on α, such that for any $\varepsilon > 0$, l, $0 < l < 1/2$, and for any solution u of equation (2), continuous up to the boundary and possessing on γ_l a normal derivative, for which the inequalities

$$|u|_S \leqslant 1; \qquad |u|_{\gamma_l} \leqslant \varepsilon; \qquad \left|\frac{\partial u}{\partial n}\right|_{\gamma_l} \leqslant \varepsilon,$$

are fulfilled, the inequality**

$$|u(0)| \leqslant C_1 \varepsilon^{C_2 l^{2+\delta}}.$$

holds.

* In the author' s paper (¹), in the analogous inequality for the Laplace equation, the author accidentally omitted the constant C_2. Namely, the inequality

$$\iint_S \operatorname{grad}^2 u[x, y, z\varphi(x, y)]\, dx\, dy < cM^{2\frac{z+1}{h}} m^{2\frac{h-z}{h}}$$

in the cited paper should be replaced by the inequality

$$\iint_S \operatorname{grad}^2 u[x, y, z\varphi(x, y)]\, dx\, dy < C_1 M^{C_2\frac{z+1}{g}} m^{C_3\frac{h-z-1}{h}}.$$

** In the paper of E. M. Landis (²), under the same restrictions, the inequality

$$|u(0)| \leqslant \varepsilon^{l^C}$$

is given, where C is a constant depending on n and α.

Theorem 3. Let G be the ball of Theorem 2, and let u be a solution of equation (2) in G, continuous together with its partial derivatives up to the boundary in a neighborhood of some point P of the boundary S of the ball G. Suppose

$$u(P) = 0, \qquad \frac{\partial u(P)}{\partial n} = 0.$$

Suppose further that the functions $u(Q)$, $\dfrac{\partial u(Q)}{\partial n}$, as the point Q tends to the point P, decrease faster than

$$e^{-(2+\delta)/|PQ|} \quad (\delta > 0)^*.$$

Then necessarily $u \equiv 0$ everywhere in G.

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CITED LITERATURE

1. M. M. Lavrent' ev, DAN, 106, No. 3 (1956).
2. E. M. Landis, DAN, 107, No. 5 (1956).

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* In paper (²), under the same restrictions on the functions $u(Q)$ and $\partial u(Q)/\partial n$, decrease faster than $e^{-C/|PQ|}$ is required, where C is a sufficiently large constant.

Note: Figure translations are in progress. See original paper for figures.

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