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1957

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Abstract

Full Text

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STABILITY IN THE LINEAR APPROXIMATION OF PERIODIC SOLUTIONS OF A SYSTEM OF DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS RIGHT-HAND SIDES

(Presented by Academician I. G. Petrovskii on 23 IV 1957)

The Lyapunov stability of a periodic solution* $z_i = \tilde{z}_i(t)$ with period τ of the system of differential equations is investigated

$$\frac{dz_i}{dt} = f_i(z_1, \dots, z_n, t). \tag{1}$$

The right-hand sides f_i are given in a certain infinite curvilinear cylinder C, whose axis is the curve $z_i = \tilde{z}_i(t)$.

Let an infinite sequence of hypersurfaces ("surfaces of discontinuity")

$$F_\alpha(z_1, \dots, z_n, t) = 0 \tag{2}$$

cut the cylinder C into domains H_α, and let the integral curve $z_i = \tilde{z}_i(t)$ intersect the hypersurfaces (2) at points M_α for $t = t_\alpha$, passing, as t increases, from the "negative" to the "positive" side of the hypersurface (2).

We shall assume that the right-hand sides f_i of equations (1) in the cylinder C satisfy the following conditions:

1°. The functions f_i are continuous in each domain H_α (including the boundaries $F_{\alpha-1} = 0$ and $F_\alpha = 0$), while, upon passage from the negative to the positive side of the hypersurface $F_\alpha = 0$, the functions f_i may have discontinuities of the first kind. The magnitude of the discontinuity of f_i at the point M_α will be denoted by ξ_i^α.

2°. The functions f_i in each domain H_α are differentiable with respect to $z_1, \dots, z_n$ at the points of the integral curve $z_i = \tilde{z}_i(t)$ $(t_{\alpha-1} \le t \le t_\alpha)$, and this differentiability is uniform with respect to t, i.e.**

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$$f_i(z_1, \dots, z_n, t) = f_i(\tilde{z}_1, \dots, \tilde{z}_n, t) + \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial z_j} \right)_{z=\tilde{z}(t)} (z_j - \tilde{z}_j) + o(\rho), \quad (3)$$

where

$$\rho = \left(\sum_{j=1}^{n} [z_j - \tilde{z}_j(t)]^2 \right)^{1/2}$$

and

$$\frac{o(\rho)}{\rho} \to 0$$

uniformly with respect to t ($t_{\alpha-1} \le t \le t_\alpha$).

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* Here and below, the Latin indices i, j take the values $1, 2, \dots, n$ (n is a fixed integer), while the Greek index α runs through the infinite sequence of values $1, 2, \dots$.

** Condition 2° is certainly satisfied if the functions f_i have continuous partial derivatives with respect to all variables in each part H_α of the cylinder C.

3°. Conditions are fulfilled which ensure the existence and uniqueness of a solution of system (1) in each domain H_α for prescribed initial values of the functions z_i, as well as conditions for the unobstructed continuous continuation of an integral curve from the domain H_α into the next domain $H_{\alpha+1}$.

4°. The functions f_i are periodic with respect to t, with period τ:

$$f_i(z_1, \dots, z_n, t + \tau) \equiv f_i(z_1, \dots, z_n, t). \quad (4)$$

Moreover, the discontinuity surfaces (2) in the cylinder C satisfy the following conditions:

5°. The hypersurfaces $F_\alpha = 0$ do not intersect one another.

6°. Each of the functions F_α is continuous, and at the points M_α is smooth; on one side of the surface $F_\alpha = 0$ one has $F_\alpha > 0$, and on the other $F_\alpha < 0$.

7°. Under a parallel shift along the t-axis by the amount τ, the family of hypersurfaces (2) goes into itself, and within the period τ the number of intersected hypersurfaces is finite.

We now define the "system of linear approximation" for system (1) with respect to the solution $z_i = \tilde{z}_i(t)$ as follows:

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1°. The integral curves of this system between the planes $t = t_{\alpha-1}$ and $t = t_\alpha$ are continuous and satisfy, in this domain, the system of linear differential equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \left(\frac{\partial f_i^\alpha}{\partial z_j}\right)_{z=\tilde{z}(t)} x_j, \tag{5}$$

where the coefficients $\left(\partial f_i / \partial z_j\right)_{z=\tilde{z}(t)}$ are computed from the functions f_i in the domain H_α.

2°. When crossing the planes $t = t_\alpha$, the integral curves $x_i = x_i(t)$ undergo discontinuities determined by the linear relations

$$x_i(t_\alpha + 0) - x_i(t_\alpha - 0) = \xi_i^\alpha \sum_{j=1}^{n} h_j^{\alpha-} x_j(t_\alpha - 0), \tag{5'}$$

where

$$h_j^{\alpha-} = \left[\frac{\partial F_\alpha / \partial z_i}{(dF_\alpha/dt)^-}\right]_{M_\alpha} \qquad \left(\frac{dF_\alpha}{dt} = \sum_{i=1}^{n} \frac{\partial F_\alpha}{\partial z_i} f_i + \frac{\partial F_\alpha}{\partial t}\right). \tag{6}$$

Let us note that, as is easy to verify, condition $(5')$ is equivalent to the condition

$$x_i(t_\alpha + 0) - x_i(t_\alpha - 0) = \xi_i^\alpha \sum_{j=1}^{n} h_j^{\alpha+} x_j(t_\alpha + 0), \tag{5''}$$

where $h_j^{\alpha+}$ is determined from (6) after replacing the minus sign in the denominator by a plus sign.

The integral curves of the nonlinear system (1) are regarded as continuous, whereas the integral curves of the system of linear approximation have discontinuities.

The significance of the system of linear approximation thus introduced is determined by the following theorem, which is an analogue of the classical results on stability by linear approximation established by Lyapunov [1] for systems of equations of the form (1) with analytic or continuous and sufficiently smooth right-hand sides.

Theorem 1. *If, under assumptions* 1°–7°, *the zero solution* $x_i = 0$ *of the system of the linear approximation (5) +* $(5')$ *is asymptotically stable, then the solution* $z_i = \tilde{z}_i(t)$ *of the original nonlinear system (1) is also asymptotically stable.*

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The course of the proof of the theorem is as follows: we write system (1) in the deviations $x_i = z_i - \tilde{z}_i(t)$, which we then transform into the variables y_i by means of the Lyapunov transformation

$$x_i = \sum_j l_{ij}(t) y_j, \tag{7}$$

where the matrix $L(t) = \|l_{ij}\|$, as in Lyapunov, is defined by the equality

$$L(t) = X(t) \cdot V^{-t/\tau}. \tag{8}$$

Here $X(t)$ is the fundamental matrix of the system of the linear approximation, and V is the constant matrix appearing in the equality

$$X(t+\tau) = X(t) \cdot V. \tag{9}$$

In the transformation (7) the coefficients $l_{ij}(t)$ are discontinuous (for $t = t_\alpha$) functions. In the variables y_i, the integral curves of the linear approximation are given by continuous functions satisfying a system of linear differential equations with constant coefficients, the same for all intervals $t_{\alpha-1} \leq t \leq t_\alpha$,

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} a_{ij} y_j, \tag{10}$$

whereas the integral curves of the original nonlinear system in the variables y_i are given by discontinuous functions $y_i = y_i(t)$ with the jump conditions

$$y_i(t_\alpha + 0) - y_i(t_\alpha - 0) = -\eta_i^{\alpha-} \sum_{j=1}^{n} g_j^{\alpha-} y_j(t_\alpha - 0), \tag{11}$$

where $\eta_i^{\alpha-}$ are related to ξ_i^α by the relations

$$\xi_i^\alpha = \sum_{j=1}^{n} l_{ij}(t_\alpha - 0) \eta_j^{\alpha-}, \tag{12}$$

and $g_j^{\alpha-}$ in the space (y, t) play the same role as the quantities $h_j^{\alpha-}$ in the space (x, t).

In what follows, the proof is carried out by means of Lyapunov functions constructed for system (10). In doing so, the jump conditions (11) are used in studying the behavior of the integral curves of the nonlinear system in the "angular" regions enclosed between the planes $t = t_\alpha$ and the corresponding surfaces of discontinuity.

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As in the classical case, the solution $x_i = 0$ of the linear approximation is asymptotically stable if and only if all the roots of the characteristic equation $\det(V - \rho E) = 0$ lie inside the unit circle $|\rho| = 1$, and is certainly unstable if at least one root lies outside this circle, i.e., if for it $|\rho| > 1$.

Theorem 2. *If at least one of the roots of the characteristic equation of the system of the linear approximation* $\det(V - \rho E) = 0$ *has modulus greater than one, then the periodic solution* $z_i = \tilde{z}_i(t)$ *of system (1) is unstable.*

The proof of Theorem 2 is analogous to the classical case, but takes into account compensation, by jumps, of a possible decrease of the function V along the trajectory in angular domains.

Remark 1. In the classical case, when the right-hand sides of equations (1) are continuous and sufficiently smooth functions, the linear approximation that decides the question of stability is written in the form (5) for the entire interval $t_0 \leqslant t < \infty$. It can be shown that also in the case under consideration, of discontinuous right-hand sides in system (1), the linear approximation for the entire interval $t_0 \leqslant t < \infty$ can be written in the form (5), if the ordinary derivative $\partial f_i/\partial z_j$ is replaced by the generalized derivative $D_j f_i$, which differs from the ordinary derivative $\partial f_i/\partial z_j$ by allowance for terms with Dirac δ-functions. Then the system of linear approximation

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \left[D_j f_i\right]_{z=\tilde{z}(t)} x_j \tag{13}$$

can be written in expanded form as follows:

$$\frac{dz_i}{dt} = \sum_{j=1}^{n} \left[\left(\frac{\partial f_i}{\partial z_j}\right)_{z=\tilde{z}(t)} + \sum_{\alpha} \xi_i^{\alpha} h_j^{\alpha}(t)\, \delta(t - t_\alpha) \right] x_j, \tag{14}$$

where

$$h_j^{\alpha}(t) = \left[\frac{\partial F_\alpha/\partial z_j}{dF_\alpha/dt}\right]_{z=\tilde{z}(t)}.$$

From this we immediately obtain (5) + (5′).

Remark 2. If the functions f_i are continuous, but their partial derivatives have discontinuities when crossing the surfaces $F_\alpha = 0$, then the discontinuity conditions (5′) disappear, and the system of linear approximation for all $t_0 \leqslant t < \infty$ is written in the usual form (5), where the coefficients $(\partial f_i/\partial z_j)_{z=\tilde{z}(t)}$ are piecewise-continuous functions having discontinuities at $t = t_\alpha$.

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Received
18 IV 1957

REFERENCES

1. A. M. Lyapunov, *The General Problem of the Stability of Motion*, Kharkov, 1892.

Note: Figure translations are in progress. See original paper for figures.

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