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Abstract

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MATHEMATICS

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DETERMINATION OF THE SMALLEST RADIUS OF UNIVALENCE FOR A CLASS OF ANALYTIC FUNCTIONS

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Let $H_1^*(a)$ denote the class of functions of the form

$$f(z)=\sum_{k=0}^{\infty} c_k z^k=\left(\sum_{k=0}^{\infty} d_k z^k\right)^2, \tag{1}$$

where the functions $F(z)=\sum_{k=0}^{\infty} d_k z^k$ are regular in $|z|<1$,

$$\sum_{k=0}^{\infty} |d_k|^2=1, \tag{2}$$

$$|c_1|=2|d_0|\,|d_1|=a \quad (0<a\leq 1). \tag{3}$$

The class $H_1^*(a)$ is a subclass of the functions of the class H_1, investigated by G. M. Goluzin (^{1}).

We shall find the smallest radius of univalence R for all functions of the class $H_1^*(a)$. For this purpose, for any fixed n we determine the smallest radius of univalence R_{2n} of all polynomials of degree $\leq 2n$ from this class, i.e., of all polynomials representable in the form

$$f_{2n}(z)=\sum_{k=0}^{2n} c_k z^k=\left(\sum_{k=0}^{n} d_k z^k\right)^2, \tag{1_n}$$

where

$$\sum_{k=0}^{n} |d_k|^2 = 1, \tag{2_n}$$

$$|c_1| = 2|d_0||d_1| = a, \tag{3}$$

and then let $n \to \infty$.

Let $f_{2n}(z) \in H_1^*(a)$. Form for it the divided difference

$$K_{2n}(z_1, z_2) = \frac{f_{2n}(z_1) - f_{2n}(z_2)}{z_1 - z_2} = \frac{(d_0 + d_1 z_1 + \ldots + d_{nz} 1^n)^2 - (d_0 + d_1 z_2 + \ldots + d_{nz} 2^n)^2}{z_1 - z_2} =$$

$$= \left[2d_0 + d_1(z_1 + z_2) + \ldots + d_n(z_1^n + z_2^n)\right]\left[d_1 + d_2(z_1 + z_2) + \ldots\right.$$

$$\left.\ldots + d_n(z_1^{n-1} + z_1^{n-2} z_2 + \ldots + z_2^{n-1})\right]. \tag{4_n}$$

It is obvious that, for $|z_1| = |z_2| = r$,

$$|2d_0 + d_1(z_1 + z_2) + \ldots + d_n(z_1^n + z_2^n)| \geq 2(|d_0| - r|d_1| - \ldots - r^n|d_n|), \tag{5_n}$$

$$\left|d_1 + d_2(z_1 + z_2) + \ldots + d_n(z_1^{n-1} + z_1^{n-2} z_2 + \ldots + z_2^{n-1})\right| \geq$$

$$\geq |d_1| - 2r|d_2| - \ldots - nr^{n-1}|d_n|, \tag{6_n}$$

and the equality signs are attained.

From the representation of the divided difference $K_{2n}(z_1, z_2)$ in the form (4_n) and from inequalities (5_n) and (6_n), it follows that

$$R_{2n} = \min\{r_n^*, r_n^{**}\}, \tag{7_n}$$

where r_n^* and r_n^{**} are, respectively, the least positive roots of the equations

$$\Phi_n^*(r) = |d_0| - r|d_1| - \ldots - r^n|d_n| = 0, \tag{8_n}$$

$$\Phi_n^{**}(r) = |d_1| - 2r|d_2| - \ldots - nr^{n-1}|d_n| = 0 \tag{9_n}$$

with respect to all $|d_0|, |d_1|, \dots, |d_n|$ satisfying the conditions (2_n) and (3).

Let first $n = 1$. It is easy to observe that the set of admissible values of the parameters $|d_0|, |d_1|$ consists in this case only of two pairs of values

$$1)\ |d_0| = \sqrt{\frac{1-\sqrt{1-a^2}}{2}}, \qquad |d_1| = \sqrt{\frac{1+\sqrt{1-a^2}}{2}};$$

$$2)\ |d_0| = \sqrt{\frac{1+\sqrt{1-a^2}}{2}}, \qquad |d_1| = \sqrt{\frac{1-\sqrt{1-a^2}}{2}}$$

(for $a = 1$ we have only one pair of values). Then from the equation

$$\Phi_1^*(r) = |d_0| - r|d_1| = 0 \tag{8_1}$$

it follows that either

$$1)\ r = \frac{1-\sqrt{1-a^2}}{a}$$

or

$$2)\ r = \frac{1+\sqrt{1-a^2}}{a}.$$

Hence it is clear that

$$R_2 = r_1^* = \frac{1-\sqrt{1-a^2}}{a}.$$

Let $n \geq 2$. To find r_n^*, we first note that the least value of the function $\Phi_n^*(r)$, for any fixed r, with respect to all $|d_k|$, $k = 0, 1, \dots, n$, satisfying the conditions (2_n) and (3), is attained only when the equalities

$$|d_{k+1}| = r|d_k|, \quad k = 2, 3, \dots, n. \tag{10_n}$$

hold. For such $|d_k|$, $k = 2, 3, \dots, n$, equation (8_n) and conditions (2_n), (3) can be rewritten in the following form:

$$|d_0| - r|d_1| - r^2 \frac{1-r^{2n-2}}{1-r^2}|d_2| = 0, \tag{11_n}$$

$$|d_0|^2 + |d_1|^2 + \frac{1 - r^{2n-2}}{1 - r^2}|d_2|^2 = 1, \tag{12_n}$$

$$|d_1| = \frac{a}{2|d_0|}. \tag{3}$$

Eliminating $|d_1|$ and $|d_2|$ from equation (11_n) by means of equalities (12_n) and (3), we obtain a biquadratic equation with respect to $|d_0|$:

$$4\left(1 + r^4\frac{1 - r^{2n-2}}{1 - r^2}\right)|d_0|^4 - 4\left(ar + r^4\frac{1 - r^{2n-2}}{1 - r^2}\right)|d_0|^2 + a^2\left(r^2 - r^4\frac{1 - r^{2n-2}}{1 - r^2}\right) = 0. \tag{13_n}$$

The problem has been reduced to determining the smallest positive root r_n^* of equation (13_n) for all possible admissible values of $|d_0|$

$$\frac{1 - \sqrt{1 - a^2}}{2} \leq |d_0|^2 \leq \frac{1 + \sqrt{1 - a^2}}{2}. \tag{14}$$

The smallest positive root r_n^* of equation (13_n) with respect to all admissible values of $|d_0|$ must be either 1) a root of equation (13_n) for

$$|d_0|^2 = \frac{1 - \sqrt{1 - a^2}}{2},$$

or 2) a root of equation (13_n) for

$$|d_0|^2 = \frac{1 + \sqrt{1 - a^2}}{2},$$

or 3) the smallest positive value $r = \tilde{r}_n$ at which the discriminant of equation (13_n) with respect to $|d_0|^2$ becomes zero.

In the first case we obtain that

$$r = \frac{1 - \sqrt{1 - a^2}}{a}.$$

In the second case we find that

$$r = \frac{1 + \sqrt{1 - a^2}}{a}.$$

In the third case we arrive at the equation

$$a^2 - 2ar + a^2r^2 - (1-a^2)r^4\frac{1-r^{2n-2}}{1-r^2} = 0.$$

This equation has the smallest positive root

$$\tilde{r}_n < \frac{1-\sqrt{1-a^2}}{a}.$$

It is not difficult to verify that, in this case, the corresponding value

$$|\tilde{d}_0|^2 = \frac{a\tilde{r}_n + \tilde{r}_n^4\dfrac{1-\tilde{r}_n^{2n-2}}{1-\tilde{r}_n^2}}{2\left(1+\tilde{r}_n^4\dfrac{1-\tilde{r}_n^{2n-2}}{1-\tilde{r}_n^2}\right)} = \frac{1}{2}\,\frac{a(a-\tilde{r}_n)}{1-a\tilde{r}_n}$$

satisfies condition (14) for all a, $0 < a \leq 1$. This means that $r_n^* = \tilde{r}_n$.

Thus, r_n^* is the smallest positive root of the equation

$$\varphi_n^*(r) = a^2 - 2ar + a^2r^2 - (1-a^2)r^4\frac{1-r^{2n-2}}{1-r^2} = 0. \tag{15_n}$$

To determine r_n^{**}, we note that, for any fixed r, the smallest value of the function $\Phi_n^{**}(r)$ with respect to all admissible values $|d_k|$, $k = 0, 1, \ldots, n$, is attained only when the condition

$$|d_{k+1}| = \frac{k+1}{k}\,r\,|d_k|, \qquad k = 2, 3, \ldots, n. \tag{16_n}$$

is fulfilled. For such values $|d_k|$, $k = 2, 3, \ldots, n$, equation (9_n) and the equalities (2_n), (3) take the following form:

$$|d_1| - \frac{|d_2|}{2}\,r\,(4 + 9r^2 + \cdots + n^2r^{2n-4}) = 0, \tag{17_n}$$

$$|d_0|^2 + |d_1|^2 + \frac{|d_2|^2}{4}\,(4 + 9r^2 + \cdots + n^2r^{2n-4}) = 1, \tag{18_n}$$

$$|d_1| = \frac{a}{2|d_0|}. \tag{3}$$

Denoting

$$4 + 9r^2 + \cdots + n^2r^{2n-4} = S_n(r)$$

and, eliminating from (17_n) $|d_1|$ and $|d_2|$ with the aid of (18_n) and (3), we obtain a biquadratic equation with respect to $|d_0|$:

$$4r^2 S_n(r)|d_0|^4 - 4r^2 S_n(r)|d_0|^2 + a^2(1 + r^2 S_n(r)) = 0, \qquad (19_n)$$

and the problem reduces to finding the least positive root r_n^{**} of equation (19_n) for all admissible values of $|d_0|$.

Repeating the arguments given above, we find that r_n^{**} is the least positive root of the equation

$$\varphi_n^{**}(r) = a^2 - (1 - a^2)r^2 S_n(r) = 0. \qquad (20_n)$$

Passing in (15_n) and (20_n) to the limit as $n \to \infty$, we obtain that the least radius of univalence R of the functions of the class $H_1^*(a)$ is determined by the equality

$$R = \min\{r^*,\ r^{**}\},$$

where r^* and r^{**} are, respectively, the least positive roots of the equations

$$\varphi^*(r) = a^2 - 2ar + 2ar^3 - r^4 = 0$$

and

$$\varphi^{**}(r) = a^2 - (4 - a^2)r^2 + 3r^4 - r^6 = 0.$$

If $R = r^*$, then the extremal function is

$$f^*(z) = \frac{e^{2i\alpha_0}}{2r^*(a - ar^{*2} + r^{*3})}\left[\frac{ar^* - (a + r^*)e^{i\alpha}z + r^{*2}e^{2i\alpha}z^2}{1 - r^*e^{i\alpha}z}\right]^2,$$

where α_0 and α are arbitrary real numbers. If $R = r^{**}$, then the extremal function is

$$f^{**}(z) = \frac{1}{2r^{**2}(4 - 3r^{**2} + r^{**4})}\left[a(1 + r^{**2})e^{i\alpha_0} + e^{i\alpha_1}\frac{r^{**2}(4 - 3r^{**2} + r^{**4})z - 2(1 + r^{**2})r^{**}e^{i\alpha}z^2 + (1 + r^{**2})}{(1 - r^{**}e^{i\alpha}z)^2}\right.$$

where α_0, α_1, and α are arbitrary real numbers. The indicated functions have zeros of the derivatives on $|z| = r^*$ and $|z| = r^{**}$, respectively.

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Note: Figure translations are in progress. See original paper for figures.