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Abstract

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THE ROLE OF THE CANCELLATION LAW IN DEFINING SEMIGROUPS WITH CANCELLATION BY MEANS OF DEFINING RELATIONS

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An associative system or semigroup is a system with one associative operation. Every semigroup can be defined by means of a system of generating elements and defining relations. We shall consider semigroups defined by a finite number of generators and defining relations. Let a system of generators be given

$$a_1, a_2, \dots, a_m. \tag{1}$$

From them we form all possible words, which are sequences of letters. We also consider the empty word. Between words an equality relation is established, determined by a system of pairs of words

$$(A_i, B_i) \qquad (i = 1, 2, \dots, n) \tag{2'}$$

according to the following rules:

1. $A_i = B_i \quad (i = 1, 2, \dots, n)$.
2. $A = A$.
3. From $A = B$ it follows that $B = A$.
4. From $A = B$ and $B = C$ it follows that $A = C$.
5. From $A = B$ it follows that $XAY = XBY$.

A, B, C, X, and Y are arbitrary words in the letters of the system (1), and A_i and B_i are the words of one of the pairs $(2')$.

Two words A and B are called equal if and only if their equality is derivable by a finite number of applications of rules 1-5. In this way the set of all words is partitioned into disjoint classes of equal words. Let α and β be two classes of equal words. By their product $\alpha\beta$ we shall mean the class of words equal to the word AB, where A is some word from the class α, and B from the class β. It follows from rules 1-5 that this operation does not depend on the choice of representatives A and B and is associative. The set of classes of equal words with this operation is precisely the associative system, or semigroup, defined by the generators (1) and the defining pairs (2′). We shall write the defining pairs (2′) in the form of defining relations

$$A_i = B_i. \tag{2}$$

If the number of defining relations (2) is zero, then the associative system (semigroup) is called the **free semigroup with generators** (1).

Let us add one more rule to rules 1-5:

6. I. From $AX = BX$ it follows that $A = B$.

Then the partition of the set of all words into classes of equal words will be different, and in the resulting semigroup the law of right cancellation will hold.

Add to 1-5 the rule:

6. II. From $XA = XB$ it follows that $A = B$.

Then we obtain a semigroup with the law of left cancellation.

We obtain a semigroup with two-sided cancellation if, to rules 1-5, we add both rule 6.I and rule 6.II.

The associative system (semigroup) with the system of generators (1) and defining relations (2) will be denoted by Π. By an **elementary transformation** of the semigroup Π we shall mean any transformation of the form

$$XA_{iY} \to XB_{iY}, \qquad XB_{iY} \to XA_{iY} \quad \text{or} \quad X \to X, \tag{3}$$

where X and Y are arbitrary words in the letters (1); $A_i = B_i$ is some defining relation from (2). It is easy to show that two words A and B in the semigroup Π will be equal if and only if there exists a sequence of elementary transformations of the semigroup Π of the form

$$A \equiv E_0 \to E_1 \to E_2 \to \cdots \to E_k \equiv B.$$

The semigroup with the law of right cancellation, given by the same generators (1) and defining relations (2), will be denoted by Π_{pr}; the corresponding semigroup with the law of left cancellation by Π_l, and the semigroup with two-sided cancellation by Π_2.

Let us also consider the following system, given by generators (1) and defining relations (2). To the system of generators (1) add new generators

$$a_1^{-1}, a_2^{-1}, \dots, a_m^{-1} \tag{4}$$

in the same number, and to the defining relations (2) the new relations

$$a_i a_i^{-1} = 1, \qquad a_i^{-1} a_i = 1 \quad (i = 1, 2, \dots, m), \tag{5}$$

where 1 denotes the empty word. As is known, the associative system defined by the generators (1) and (4) and the defining relations (2) and (5) will be a group. We shall call this group the group given by the generators (1) and defining relations (2), and denote it by Γ. Obviously, if two words A and B are equal in the semigroup Π, then they are equal also in the systems Π_{pr} and Π_l. If two words are equal in Π_{pr} or Π_l, then they are equal in Π_2 and in Γ. This circumstance may be represented by the following diagram:

$$\begin{tikzcd}
 & \Pi_{\mathrm{pr}} \arrow[dr] & & \\
\Pi \arrow[ur] \arrow[dr] & & \Pi_2 \arrow[r] & \Gamma. \\
 & \Pi_l \arrow[ur] & &
\end{tikzcd}$$

The free systems Π, Π_{pr}, Π_l, and Π_2 coincide.

An associative system with three generators a, b, c and two defining relations $ab = ac$ and $ba = ca$ cannot be isomorphic to any system Π_{pr}, Π_l, or Π_2. It is also not difficult to show that the system Π_{pr} (Π_l) with one defining relation $ab = ac$ ($ba = ca$) cannot be isomorphic to any system Π_2.

If the number of defining relations is not restricted, then every system Π_{pr}, Π_l, or Π_2 can be given as a system Π, i.e. without adding rules 6.I and 6.II (in this case it may be necessary to add an infinite number of defining relations).

Let us consider systems Π_2 given by a finite number of generators and defining relations. The question arises whether each of these systems can be given as a system Π with a finite number of defining relations. The answer to this question is given by the following theorem.

Theorem 1. *There exists a semigroup with cancellations* Π_2*, given by a finite number of generators and defining relations, which is not isomorphic to any semigroup* Π *with a finite number of defining relations.*

An analogous problem can be posed for the systems Π_{pr} and Π_{l} with one-sided cancellations, and in this case the corresponding theorem is true.

Theorem 2. *There exists a semigroup with the law of right-sided (left-sided) cancellation, given by a finite number of generators and defining relations, which is not isomorphic to any semigroup* Π *with a finite number of defining relations.*

Theorem 3. *There exists a semigroup* Π_2 *with cancellations, given by a finite number of generators and defining relations, which is not isomorphic to any semigroup* Π_{l} (Π_{pr}) *with a finite number of defining relations.*

Theorems 1, 2, and 3 assert that, in defining semigroups with cancellation, rules 6.I and 6.II cannot be replaced by a finite number of defining relations, even if one preserves one of these rules, which are implications involving variables (A, B, X). The question arises: can they not be replaced by a finite number of defining relations and identical equalities. In a given system of words, we shall call a **mixed relation** any equality containing both variable symbols (replaceable by any word of the system under consideration) and concrete words of this system. In particular, every defining relation and every identical equality will be a mixed relation. The following theorems assert that rules 6.I and 6.II cannot be replaced by a finite number of mixed relations*.

Theorem 4. *There exists a semigroup with cancellations* Π_2*, given by a finite number of generators and defining relations, which is not isomorphic to any system* Π*,* Π_l*, or* Π_pr *with a finite number of mixed relations.*

Theorem 5. *There exists a semigroup with the law of right-sided (left-sided) cancellation, given by a finite number of generators and defining relations, which is not isomorphic to any system* Π *with a finite number of mixed relations.*

Let the semigroup given by the generators

$$p_1,\ p_2,\ a,\ b,\ \nu,\ \mu,\ t_1,\ t_2 \tag{6}$$

and the defining relations

$$p_1t_1 = \nu p_1\mu, \qquad \mu b = b\mu, \qquad \mu p_2 = p_2\mu t_2, \qquad p_2t_2 = t_1p_2, \qquad at_1 = t_1b, \tag{7}$$

be denoted by $\Pi^{(0)}$. The systems Π_pr, Π_2, and Γ, given by the same generators and defining relations, will be denoted by $\Pi_\mathrm{pr}^{(0)}$, $\Pi_2^{(0)}$, and $\Gamma^{(0)}$. Note that the corresponding system $\Pi_\mathrm{l}^{(0)}$ is isomorphic to $\Pi^{(0)}$.

Lemma 1. *In the semigroup* $\Pi_2^{(0)}$ *the equalities*

$$p_1a^kp_2 = \nu p_1b^kp_2\mu \qquad (k = 1, 2, \ldots) \tag{8}$$

hold.

Proof. We have the following sequence of elementary transformations of the semigroup $\Pi^{(0)}$:

$$p_1a^kp_2t_2 \to p_1a^kt_1p_2 \to \cdots \to p_1t_1b^kp_2 \to \nu p_1b^k\mu p_2 \to \cdots$$

$$\cdots \to \nu p_1\mu b^kp_2 \to \nu p_1b^kp_2\mu t_2.$$

* Theorems 4 and 5 will also remain true if, as mixed relations, we allow the addition of implications of equalities not containing variable symbols. For example: $A_0 = B_0 \to C_0 = D_0$, where A_0, B_0, C_0, and D_0 are certain words of the system under consideration.

From the equality $p_1 a^k p_2 t_2 = \nu p_1 b^k p_2 \mu t_2$, applying rule 6.I, we find
$p_1 a^k p_2 = \nu p_1 b^k p_2 \mu$ in $\Pi^{(0)}_{\mathrm{pr}}$.

Corollary. The equalities (8) hold in the semigroup $\Pi^{(0)}_{\mathrm{pr}}$. In deriving each of the equalities (8), rule 6.I is applied only once.

Lemma 2. *Every word composed of positive powers of the letters (6) and equal in the group $\Gamma^{(0)}$ to the word $p_1 a^k p_2$ is either the word $p_1 a^k p_2$ itself, or $\nu p_1 b^k p_2 \mu$.*

We omit the proof of Lemma 2 because of its complexity. On the basis of these lemmas it is easily proved that the semigroup $\Pi^{(0)}_2$ is the desired semigroup Π_2, whose existence is asserted by Theorem 1. Suppose that the semigroup $\Pi^{(0)}_2$ is isomorphic to some semigroup Π', given by a finite number of defining relations

$$C_j = D_j \quad (j = 1, 2, \dots, \lambda), \tag{9}$$

but without the rules 6.I and 6.II*. Then, by Lemma 1 and by virtue of the isomorphism, all equalities (8) must hold in Π'. Each of them must be derivable by some sequence of elementary transformations of the semigroup Π':

$$p_1 a^k p_2 \equiv E_0 \to E_1 \to \cdots \to E_{m_k} \equiv \nu p_1 b^k p_2 \mu.$$

Every equality holding in the semigroup Π' must hold in $\Pi^{(0)}_2$, and hence also in $\Gamma^{(0)}$. Consequently, all the words $E_0, E_1, \dots, E_{m_k}$ are equal to the word $p_1 a^k p_2$ in the group $\Gamma^{(0)}$ and, moreover, contain only positive powers of the letters (6). By Lemma 2, each of these words is either $p_1 a^k p_2$, or $\nu p_1 b^k p_2 \mu$. This means precisely that among the defining relations (9) of the semigroup Π' there must occur the equality

$$p_1 a^k p_2 = \nu p_1 b^k p_2 \mu,$$

where k is any natural number. The latter is incompatible with the condition that the number of defining relations (9) is finite.

Theorem 2 is proved analogously. Theorem 3 is somewhat more difficult. In the left-sided case one considers the semigroup $\Pi^{(1)}_l$ with defining relations

$$t_2 p_1 = p_1 t_1, \qquad t_1 a = b t_1, \qquad t_1 p_2 = \nu p_2 \mu, \qquad b\nu = \nu b, \qquad p_1 \nu = t_2 \nu p_1.$$

Theorems 4 and 5 are easily derived from Theorems 1, 2, and 3 on the basis of the following lemma.

Lemma 3. *Every mixed relation holding in the semigroup* $\Pi_2^{(0)}$ *has the form*

$$A_1 x_1 A_2 x_2 \cdots A_k x_k A_{k+1} = B_1 x_1 B_2 x_2 \cdots B_k x_k B_{k+1},$$

where $x_1, x_2, \dots, x_k$ *are variable symbols;* A_i, B_i *are certain words of the semigroup* $\Pi_2^{(0)}$*, and* $A_i = B_i$ *in* $\Pi_2^{(0)}$.

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* Note that, in view of the isomorphism of $\Pi_2^{(0)}$ and Π', the same letters of the system (6) may serve as generators of the semigroup Π'.

Note: Figure translations are in progress. See original paper for figures.