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Abstract

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APPLICATION OF THE THEORY OF CONTINUOUS GROUPS OF TRANSFORMATIONS TO THE STUDY OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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§ 1. Consider an equation of the form

$$\frac{dy}{dx} = \frac{Y(x,y)}{X(x,y)}, \tag{1}$$

where $Y(x,y)$, $X(x,y)$ are rational integral functions of x, y.

The problems arising in the study of solutions of a differential equation of the indicated form are divided into two types: the study of integral curves in a neighborhood of a singular point (the local problem) and the study of the arrangement of integral curves as a whole. In solving the second problem, an important role is played by trajectories satisfying special conditions, for example those passing through singular points of saddle type, singular solutions, limit cycles, etc.

Of chief interest is the study of limit cycles. The determination of cycles, as has been shown by the works of Mandelstam, Papaleksi, Andronov, and many other scholars in our country and abroad, plays an important role in various applications of the qualitative theory. The study of limit cycles of equation (1) and, in the more general case, the study of periodic solutions of a system of equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n}$$

has been the subject of numerous investigations.

The principal results, beginning with the fundamental works of Poincaré and Lyapunov, obtained in this area up to 1949, are set forth in ([1], [2]). A detailed survey of the later literature, together with a bibliography, is given in ([3]). Among works that have appeared recently, one should note ([4]), in which estimates are given for the number of limit cycles in a neighborhood of a singular point, and the recently published work ([5]). In ([5]) a method is proposed for determining the number of limit cycles, distinct from the previously known methods of Poincaré and Lyapunov, and by means of this method the problem of determining the number of cycles of equation (1) is solved for the case when Y and X are polynomials of the second degree.

§ 2. We shall show that, in determining the limit cycles of equation (1), the theory of continuous groups can be applied. Let X and Y be polynomials of degree n with real coefficients. In the plane $\operatorname{Im} x = \operatorname{Im} y = 0$, take an arbitrary domain D, bounded by a convex curve. Denote the operator $X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} = A$, and consider the equation

$$AM = \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) M. \tag{2}$$

Let us formulate the properties of the solution of equation (2) that will be needed below. (The size of the present article does not permit giving their proofs.)

Theorem 1. There always exists a solution of equation (2) of the form

$$M = e^{\Theta_1(x,y)|\Theta_2(x,y)},$$

where $\Theta_1(x, y)$, $\Theta_2(x, y)$ satisfy the condition: for every point $P(a, b) \subset D$ there exists $r = r(a; b) > 0$ such that in the domain $|x - a| < r;\ |y - b| < r$ the functions Θ_1, Θ_2 are holomorphic.

Theorem 2. The functions Θ_1, Θ_2 can always be chosen so that in the domain $D' = D - \sum P_k$, where P_k are the singular points of equation (1), the function M will be single-valued and holomorphic.

Theorem 3. If the singular points of equation (1) are of the first multiplicity, then one can always find a function M, single-valued and holomorphic in the domain D' and continuous in the domain D.

From the existence of the function M, analytic in $D - \sum P_k$, it follows that there exist components ξ and η of an infinitesimal transformation of the group of the equation $X\,dy - Y\,dx = 0$.

Solving the system of equations

$$X\eta - Y\xi = M, \qquad (aX + bY)\eta + (\alpha X + \beta Y)\xi = N, \tag{3}$$

where N, a, b, α, β are arbitrary continuous functions satisfying the condition

$$\beta^2 + 2\beta a + a^2 - 4ab < 0 \quad \text{for } x, y \subset D, \tag{4}$$

we obtain for ξ and η values continuous in $D - \sum P_k$. Indeed, if condition (4) is fulfilled, then the determinant of system (3) in $D - \sum P_k$ is not equal to zero. It is easy to verify that for singular points of first multiplicity there exist ξ and η continuous in D.

§ 3. The expressions obtained for M may be used in the study of solutions of the equation $Y\,dx - X\,dy = 0$.

Consider the problem: to find an upper bound for the number of limit cycles of the equation $Y\,dx - X\,dy = 0$.

For its solution we shall use lemmas.

Lemma 1. Consider the domain $D - \sum P_k$ and a cycle lying entirely in D. Let ξ and η be the components of a transformation of the group of the equation, continuous and single-valued in $D - \sum P_k$. Since under this transformation a cycle cannot pass into another curve, it is clear that on the cycle

$$X\eta - Y\xi = 0. \tag{5}$$

Lemma 2. Consider the equation

$$X\frac{\partial u}{\partial x} + Y\frac{\partial u}{\partial y} = \Theta(x, y). \tag{6}$$

It is obvious that for any functions X, Y, defined in the domain D, one can always find $\Theta(x, y)$ for which equation (5) has an integral in the domain D.

For what follows, Lemma 3 is important.

Lemma 3. If X, Y are polynomials of degree n having singular points of first multiplicity, then as $\Theta(x, y)$ one may choose any function satisfying the conditions: 1) $\Theta(x, y)$ is an analytic function in D; 2) at the singular points of the equation $dy/dx = Y/X$, the function Θ has a zero of the same multiplicity as X, Y.

Indeed, equation (6) has an integral analytic in the domain $D - \sum P_k$. For this it is sufficient to take an analytic curve passing through all singular points of the equation, and to prescribe on it, as initial values of u, the values of a bounded analytic function,

for example, $u = \text{const}$. In the domain $D - \sum P_k$, equation (6) can always be regarded as an equation satisfying the conditions of the Cauchy-Kovalevskaya theorem. We shall show that this integral can be continued to the singular points and is single-valued in $D - \sum P_k$. Indeed, the increment of the integral u in going around a singular point is, as is easy to see,

$$\lim_{\rho\to 0}\oint \frac{\partial u}{\partial x}\,dx+\frac{\partial u}{\partial y}\,dy, \tag{7}$$

where the integral is taken over a circle with center at the singular point.

Instead of a circle, it is, of course, possible to take any closed curve without self-intersections. Such curves are easy to construct for a center and a focus. For a center it is enough to take an integral curve of equation (1); for a focus, a curve composed of an integral curve of equation (1) (a spiral) and an arc of a curve on which the value of u is prescribed. On these curves $du/ds = \Theta/\sqrt{x^2+y^2}$, i.e. a bounded quantity (here s is the arc length of the curve). From the boundedness of du/ds it follows that (7) is equal to zero. For singular points corresponding to real roots of the characteristic equation of the differential equation (1), u is represented in the form of a power series. The partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ are bounded in a neighborhood of the singular point, and expression (7) is equal to zero.

§ 4. We now turn to the determination of the maximum number of limit cycles of the equation. Let the equation have a cycle $\Phi_k(x,y)=0$, where Φ_k is an analytic function defined in D. Then $A\Phi_k=\lambda_k(x,y)\Phi_k$, where $\lambda_k(x,y)$ is an analytic function defined in D. If μ_k is an analytic function, then

$$Ae^{\mu_k}\Phi_k=(A\mu_k+\lambda_k)e^{\mu_k}\Phi_k. \tag{8}$$

Consider the equation

$$A\mu_k+\lambda_k=P_k(x,y), \tag{9}$$

where P_k is a polynomial determined by the condition that $P_k-\lambda_k$ vanishes at all singular points of the equation $dy/dx=Y/X$. If Φ_k is known, then one can find λ_k, and, consequently, find P_k. Having computed P_k, by the preceding one can find μ_k in the whole domain D. Since the polynomial P_k must assume prescribed values at n^2 singular points, the degree n_k of the polynomial P_k is determined from the condition that n_k is the least integer greater than or equal to the positive root of the equation

$$n_k^2+3n_k+2-2n^2=0, \tag{10}$$

where n is the degree of the polynomials X and Y.

The indicated bound for the degree is obtained under the assumption that all equations for determining the coefficients of the polynomial P_k are independent. Since the singular points lie at the intersections of the curves $X=0$, $Y=0$, these equations will not be independent. Indeed, if the polynomial P_k assumes the required values at the singular points, then the polynomial

$$P_k + s(x,y)X(x,y) + q(x,y)Y(x,y) = Q_k(x,y) \tag{11}$$

assumes at the singular points the same values for arbitrary functions s and q. Let s and q be polynomials of degree $n_k - n$. From formula (11) we obtain an upper bound for the number of limit cycles of the equation.

Suppose that the equation $X\,dy - Y\,dx = 0$ has N limit cycles. The equations of these cycles are $F_l(x,y) = 0$, where $l = 1, 2, \dots, N$. For each cycle the following equality holds:

$$AF_l = \lambda_l F_l. \tag{12}$$

For each cycle F_l let us find a multiplier e^{μ_l} such that $Ae^{\mu_l}F_l = A\Phi_l = Q_l\Phi_l$, where Q_l is a polynomial. By the preceding, such a function μ_l always exists.

Denoting $\prod \Phi_l^{\alpha_l} = \psi$, where α_l are certain numbers, by formula 4 we obtain:

$$A\psi = \left(\sum_{l=1}^{N} \alpha_l Q_l\right)\psi = \left[\sum_{l=1}^{N} \alpha_l P_k^l + sX + qY\right]\psi,$$

where s and q are arbitrary polynomials of degree $n_k - n$. Having computed, by formula (10), the upper bound for the degree n_k and having determined, by Noether' s theorem [6], the number of independent parameters in the polynomial $sX + qY$, one can solve the following problem.

Find such a number N that the identity

$$\sum_{l=1}^{N} \alpha_l P_k^l + sX + qY = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}. \tag{13}$$

holds.

Equating the coefficients of like powers in the left- and right-hand sides, we obtain $(n_k+1)(n_k+2)/2$ equations, where n_k is the positive root of equation (10). Hence N (the number of parameters) must be equal to

$$N = (n_k+1)(n_k+2)/2 - N_1,$$

where N_1 is the number of independent parameters of the polynomial $s(x,y)X(x,y) + q(x,y)Y(x,y)$. The number N_1 can always be computed by Noether' s theorem. In the case of points of first multiplicity it is easy to find that*

$$N \le (2\sqrt{2} - 2)n^2.$$

Let us prove that the number of cycles cannot exceed N. Indeed, if relation (13) holds, then $\psi = \prod \Phi_l^{\alpha_l} = M$, and M must vanish on the limit cycles. It follows that there are no cycles different from the cycles $F_l = 0$, $l = 1, 2, \dots, N$.

If there exist $N+1$ closed integral curves, then at least one of them cannot be invariant under the transformation of the group of equation (1), and consequently there exists a continuum of closed curves.

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* N_1 is equal to the number of independent arbitrary parameters of the polynomial $sX + qY$, since s and q are polynomials of degree $n_k - n$. Determining n_k, we obtain $N_1 \approx (\sqrt{2} - 1)^2 n^2$ up to terms of order n.

Note: Figure translations are in progress. See original paper for figures.